Manipulation and Single-Peakedness: A General Result

Elizabeth Maggie Penn   Washington University in Saint Louis
John W. Patty   Washington University in Saint Louis
Sean Gailmard   University of California at Berkeley

This article considers environments in which individual preferences are single-peaked with respect to an unspecified, but unidimensional, ordering of the alternative space. We show that in these environments, any institution that is coalitionally strategy-proof must be dictatorial. Thus, any nondictatorial institutional environment that does not explicitly utilize an a priori ordering over alternatives in order to render a collective decision is necessarily prone to the strategic misrepresentation of preferences by an individual or a group. Moreover, we prove in this environment that for any nondictatorial institution, the truthful revelation of preferences can never be a dominant strategy equilibrium. Accordingly, an incentive to behave insincerely is inherent to the vast majority of real-world lawmaking systems, even when the policy space is unidimensional and the core is nonempty.

The question of *manipulation*—the misrepresentation of one's true preferences in order to achieve a more favorable collective decision—has alternately intrigued and frustrated social scientists for over two centuries.[1] Manipulation is clearly a potential problem for those interested in drawing inferences from individuals' observed behaviors. For example, how does a legislator's roll-call vote relate to his or her policy preferences? How does an individual voter's vote choice reflect his or her preferences over the parties and/or candidates? More subtly, does the composition of a legislative committee reflect the preferences of the members of the legislature? Does a juror's vote to convict a defendant truthfully reflect the juror's beliefs about the defendant's guilt or innocence? Is the president's choice of political appointees indicative of the policies that he or she would pursue in their positions? In short, manipulation is simultaneously a methodological and normative conundrum of broad relevance to those who study politics.

Furthermore, the conundrum is—in a very precise sense—inescapable: Allan Gibbard and Mark Satterthwaite famously and independently demonstrated that, when there are at least three alternatives to choose from, any collective choice procedure that eliminates the potential incentive for manipulation must be dictatorial.[2] The Gibbard-Satterthwaite theorem closely mirrors Kenneth Arrow's impossibility result, which states that any democratic institution that can aggregate individual preferences so as to always produce a transitive social ranking of the alternatives without violating either Pareto efficiency or independence from irrelevant alternatives (IIA) must also be dictatorial.[3] The theorems of Arrow, Gibbard, and Satterthwaite all have strongly negative implications about the predictability and performance of

[1]Illustrating the topic's long lineage is Condorcet's 1788 treatise "On the Constitution and the Functions of Provincial Assemblies," in which he argued that Borda's scoring method was highly prone to strategic manipulation by voters (cf. Young 1995).

[2]See Gibbard (1973) and Satterthwaite (1975).

[3]See Arrow (1963).

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democratic group choice. Perhaps the most famous argument along these lines was offered by Riker (1982).[4] It is indeed accurate to summarize the perception of democratic governance following these seminal results as "bedevilled by impossibility, instability, and arbitrariness" (Dryzek and List 2003, 2).

On the other hand, Black's median voter theorem implies that Arrow's negative conclusions regarding preference aggregation disappear if one presumes that the policies can be ordered along a left-right spectrum so that every individual's preference over the alternatives is single-peaked.[5] The assumption that individuals have single-peaked preferences guarantees the existence of at least one alternative that cannot be defeated by any other under majority rule (i.e., the majority rule core is nonempty). Furthermore, the importance of this result for the development of political theory over the past half century is hard to overstate: contrasted with the results of McKelvey (1976), Schofield (1978), Rubinstein (1979), and Riker (1980), the theoretical power of the unidimensional spatial model underlying the institutional theories of the past 30 years has proven undeniably productive in advancing our understanding of the interactions between individual preferences, strategic behavior, and political institutions.[6] Indeed, the central insight of the notion of structure-induced equilibrium (Shepsle 1979) is that institutions might be structured so as to reduce a single, difficult-to-predict multidimensional policy decision into a well-ordered set of unidimensional, partial policy decisions. The following quotes demonstrate that the distinction between the unidimensional and multidimensional settings has been repeatedly cast as dispositive with respect to, variously, the simplicity, stability, predictability, and even coherence of political decision making.

> "When all individuals have single peaked preference orderings the process of collective decision making is dramatically simplified." (Feld and Grofman 1988, 776)

> "Where preferences are single peaked . . . , one option must be the Condorcet winner and it

would be possible to find this by repeated binary votes." (Miller 1992, 63)

> "In a unidimensional world, there is no incentive for 'sophisticated voting.'" (Mouw and Mackuen 1992, 102, fn. 4)

> "[When preferences are single peaked] voters know the political world is coherently organized, the possibility of cycles is zero and the method of majority rule is wholly consistent and never tyrannical." (Riker 1992, 107)

> "[R]elaxation of [unrestricted domain] provides acceptable escape-routes from Arrow's theorem and from the Gibbard-Satterthwaite theorem, compatible with all other conditions of these theorems." (Dryzek and List 2003, 27–28)

> "We do know for sure that if the distribution of preference orders is such that they are single peaked, the Gibbard-Satterthwaite theorem does not apply, there is no chance for strategic voting to succeed." (Mackie 2003, 161)

In this article, we distinguish between the implications of single-peakedness for Arrow's result and the implications of this assumption for the Gibbard-Satterthwaite theorem. Specifically, while single-peakedness is sufficient to escape from the negative conclusions of Arrow, we demonstrate in this article that the assumption of single-peaked preferences is not sufficient to eliminate the possibility of profitable manipulation of democratic collective choice. That is, when there are at least three alternatives to choose from, the potential for strategic behavior is endemic to all democratic political institutions, even when preferences are single-peaked.[7]

Our results rest upon the fact that, in order for single-peaked preferences to provide an escape route from the conclusions of Gibbard-Satterthwaite, one must require that individuals may vote for alternatives *only in a way that is consistent with the underlying ordering of alternatives.* In other words, if the underlying ordering of alternatives on the left-right spectrum is $x < y < z$, then ruling out profitable manipulation requires that one disallow any

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[4]Of course, Riker's arguments have—quite rightfully—not gone unchallenged. While they adopt very different starting points, both Mackie (2003) and McGann (2006) have recently offered counterarguments to Riker's criticisms of democratic institutions.

[5]Black (1948).

[6]This literature is too vast to cite in anywhere near an exhaustive fashion, but a list of seminal contributions in this literature would certainly include Romer and Rosenthal (1979), Denzau and Mackay (1983), Austen-Smith and Banks (1988), Gilligan and Krehbiel (1989, 1990), Krehbiel (1991, 1998), Rohde (1991), Cox and McCubbins (1993, 2005), Epstein and O'Halloran (1999), and Groseclose and McCarty (2001).

[7]It is almost obligatory to note at this point that—in spite of its potentially unsettling mendaciousness—"manipulation" need not be a bad thing. For example, Miller (1977) shows that when all individuals vote strategically, outcomes may be obtained that are Pareto superior to outcomes obtained if all individuals vote truthfully. Furthermore, "manipulation" is a large tent, encompassing a wide array of behaviors. For example, Dowding and Van Hees (2007) distinguish between "sincere" and "nonsincere" forms of manipulation.

individual, for example, from casting a vote for x over z and a vote for z over y. In other words, Black's theorem implies that majority rule is strategy-proof provided that individuals aren't *too* strategic.[8] In this sense, our results illustrate that, even with a strong restriction on the structure of the underlying preferences, "institutions matter" insofar as every democratic procedure necessarily generates the incentive for strategic misrepresentation by one or more individuals.

Thus, the arguments in this article demonstrate that Riker's famous—and arguably unwarranted—dismissal of single-peakedness as empirically irrelevant[9] was unnecessary for his conclusion that political outcomes "are the consequence not only of institutions and tastes, but also of the political skill and artistry of those who manipulate agenda, formulate and reformulate questions, generate 'false' issues, etc., in order to exploit the disequilibrium of tastes for their own advantage" (Riker 1980, 445). Riker's discussion of political strategy—particularly the introduction of "false issues"—is directly spoken to by our results. This is because one of the keys to our result is that knowing with certainty that the alternatives *can* be ordered so as to induce a single-peaked profile of preferences is not equivalent to knowing *how* the alternatives will be ordered. As we demonstrate in both the proof of the result and two motivating examples following it, the assumption of single-peaked preferences is not sufficient to rule out some situation existing in which a group of individuals agree that they would prefer to misrepresent (or, perhaps, more charitably, "dispute") the true left-right ordering of alternatives. Thus, referring to the passage from Riker (1992) quoted earlier, our results demonstrate while the presumption of single-peaked preferences does imply that collective preference is "coherently organized," and the possibility of *sincere majority preference* cycles is indeed zero, one should still not neglect political institutions, as any democratic choice procedure necessarily holds the possibility of a *revealed* collective preference cycle.

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[8]This point was—to our knowledge—first noted by Blin and Satterthwaite (1976), who examined one specific collective choice rule (majority rule with Borda completion) and showed that it is strategy-proof when preferences and ballots are required to be single-peaked with respect to a common ordering, but is manipulable when ballots are no longer required to be single-peaked with respect to the common ordering. Our results are stronger in one sense—we consider *all* collective choice rules—and somewhat qualified in another—we focus on coalitional strategy-proofness, which is a stronger condition for a collective choice rule to satisfy. In the canonical special cases of exactly three voters, our results are strict generalizations of the insights of Blin and Satterthwaite.

[9]Riker (1980, 438).

Notation and Definitions

We consider choice from a finite collection of K *alternatives* (or *policies*), X, and a finite collection of n *individuals* (or *voters*) N. We assume that $K \geq 3$ and $n \geq 2$. Individual i's preferences are represented by a strict, transitive, and complete binary relation P_i. The notation $x P_i y$ implies that i strictly prefers x to y. If $x_i^* P_i y$ for all $y \neq x_i^*$, then x_i^* is referred to as i's *most-preferred policy* or *ideal point*.

Throughout, $\rho = (P_1, \ldots, P_n)$ denotes an n-dimensional preference profile describing the preferences of all individuals, and we let $x_i^*(\rho)$ denote i's ideal point at profile ρ. The notation $\mathcal{P}^n$ represents the collection of all n-dimensional profiles of strict orders on X. Any nonempty set $\mathcal{D} \subseteq \mathcal{P}^n$ is referred to as a *preference domain*, and when $\mathcal{D}$ is a strict subset of $\mathcal{P}^n$ then $\mathcal{D}$ is referred to as a *restricted domain*. We discuss restricted domains in more detail in a following section. For any preference profile $\rho \in \mathcal{P}^n$, $\rho|_S$ denotes the restriction of ρ to the set of alternatives $S \subseteq X$. Similarly, for any individual preference $P_i \in \mathcal{P}$, $P_i|_S$ denotes the restriction of i's preference relation to the set S. For any preference profile $\rho \in \mathcal{P}^n$ and pair of alternatives $(x, y) \in X^2$, the notation $P(x, y; \rho) \equiv \{i \in N: x P_i y\}$ denotes the set of individuals who strictly prefer x to y under ρ.

Collective Choice Functions

A *collective choice function*, or *choice function*, is any function, $\phi : \mathcal{P}^n \to X$ that maps any strict profile of orderings over alternatives into the policy space X. Throughout, we will assume that ϕ has *full range*: for any $x \in X$, there exists a $\rho \in \mathcal{P}^n$ such that $\phi(\rho) = x$. In words, the notation $\phi(\rho) = x$ means that the choice function ϕ chooses outcome $x \in X$ when the profile of individuals' preferences is given by ρ.

While we require choice functions to map any strict profile into a social outcome, we do not require individuals' true preference orderings to be drawn from the full set $\mathcal{P}^n$. This is because we are interested in the (possibly insincere) behavior induced by a choice function when true individual preferences are drawn from a restricted domain. We call the *preference domain* of a choice function $\mathcal{D}$, while the *ballot domain* of all choice functions is assumed to be $\mathcal{P}^n$. In other words, while true individual preferences may come from a restricted set of orderings, individual behavior is only required to appear individually rational, in the sense of being rationalizable by a transitive binary relation. Throughout, we will use the notation (P_i', ρ_{-i}) to denote a ballot profile in which i

submits ballot P'_i, and all others submit ballots as in profile ρ. More generally, the notation (ρ'_L, ρ_{-L}) denotes a ballot profile in which all members $i \in L \subseteq N$ submit ballots as under ρ', and all individuals not in L submit ballots as under ρ.

The following definitions characterize several properties of collective choice functions.

Definition 1 (Weakly Paretian). *A collective choice function* ϕ *is* weakly Paretian *on* $\mathcal{D}$ *if for all* $\rho \in \mathcal{D}$ *and all* $(x, y) \in X^2$,

$$P(x, y; \rho) = N \Rightarrow \phi(\rho) \neq y.$$

Definition 2 (Monotonic). *A collective choice function* ϕ *is* monotonic *on* $\mathcal{D}$ *if, for all* $(x, y) \in X^2$ *and all* $\rho, \rho' \in \mathcal{D}$,

$$P(x, y; \rho) \subseteq P(x, y; \rho') \text{ and } \phi(\rho) = x \Rightarrow \phi(\rho') \neq y.$$

Definition 3 (Dictatorial). *A collective choice function* ϕ *is* dictatorial *if for some* $i \in N$ *and for all* $\rho \in \mathcal{P}^n$,

$$\phi(\rho) = x_i^*(\rho),$$

where $x_i^*(\rho)$ *is* i*'s reported ideal point under ballot profile* ρ.[10]

Definition 4 (Strategy-Proof [SP]). *A collective choice function* ϕ *is* manipulable *if, for some* $\rho = (P_1, \ldots, P_n) \in \mathcal{D}$ *and* $i \in N$ *there exists a* $P'_i \in \mathcal{P}$ *such that*

$$\phi(P'_i, \rho_{-i}) P_i \phi(\rho).$$

A choice function is strategy-proof *if it is not manipulable.*

Definition 5 (Coalitionally Strategy-Proof [CSP]). *A collective choice function* ϕ *is* coalitionally manipulable *if, for some* $\rho = (P_1, \ldots, P_n) \in \mathcal{D}$ *and* $L \subseteq N$ *there exists a* $\rho' \in \mathcal{P}^n$ *such that*

$$\phi(\rho'_L, \rho_{-L}) P_i \phi(\rho) \text{ for all } i \in L.$$

A choice function is coalitionally strategy-proof *if it is not coalitionally manipulable.*

Note that individual manipulability of social choice functions implies coalitional manipulability (for a coalition of one); equivalently, coalitional strategy-proofness implies strategy-proofness. The converse is not generally true; there may be instances in which a social choice

function may be manipulable by a sufficiently large coalition, but not by an individual. However, when $\mathcal{D} = \mathcal{P}^n$, then individual strategy-proofness implies coalitional strategy-proofness, because it implies dictatorship. Thus, in a setting with unrestricted domain, individual and coalitional manipulability and nonmanipulability are equivalent.

Single-Peaked Preferences

In this section, we define the domain of single-peaked preferences. This domain has attracted the interest of many scholars because it has been shown to lead to the existence of nondictatorial Arrovian preference aggregation rules and, when ballots are required to be single-peaked, to nonmanipulable, nondictatorial collective choice functions. Our interest is less about existence, and more about the characterization of choice functions on this restricted domain.

Single-Peaked Preferences. A preference profile is *single-peaked* if there exists a way of ordering the collection of alternatives along a left-right scale so that each individual's ranking of the alternatives decreases as one moves away from his ideal point (Austen-Smith and Banks 1999, 93). We denote the single-peaked preference domain by $\mathcal{S} \subset \mathcal{P}^n$. We will at times denote by Q_ρ a particular ordering of alternatives with respect to which profile $\rho \in \mathcal{S}$ is single-peaked. When referring to an ordering Q, if alternative x is above y with respect to Q we write $x Q y$.

While the single-peaked preference restriction is widely utilized and intuitively quite simple, Ballester and Haeringer (2011) prove that the set $\mathcal{S}$ is completely characterized by two conditions, worst-restriction[11] and α-restriction, each of which is defined below. Prior to that, however, it is important to emphasize that the domain $\mathcal{S}$ is the set of *all* single-peaked preference profiles. In other words, in a priori terms, any ordering of the alternatives is possible.[12]

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[10]Note that the previous definitions are defined on the preference domain $\mathcal{D}$ while dictator is defined on ballot domain $\mathcal{P}^n$. We make these definitional choices conservatively; we are ultimately going to prove that coalitional strategy-proofness when preferences are single-peaked will imply dictator on the full ballot domain. Our intermediary steps in this proof will only utilize the weak Pareto and monotonicity properties on the restricted preference domain. However, by implying dictator on the full ballot domain, our proof also guarantees that the collective choice functions we consider are both weakly Paretian and monotonic on the full ballot domain.

[11]See Sen (1966) and Sen and Pattanaik (1969) for a more thorough discussion of worst-restriction.

[12]This point is a technical one, but important for broader considerations of the results in this article. In particular, for any given linear ordering of the alternatives, $Q \in \mathcal{P}$, one can identify the set of preferences that are single-peaked with respect to $\mathcal{Q}$, this set is denoted by $\mathcal{S}_Q$, and the set of all profiles of such preferences is denoted by $\mathcal{S}_Q^n$, a rectangular (product) space. Conversely, the set $\mathcal{S}$ that we consider is not rectangular. Unlike $\mathcal{S}$, the space $\mathcal{S}_Q^n$ is widely discussed in the political economy literature. In particular, it has been shown that in domain $\mathcal{S}_Q^n$ the collection of strategy-proof collective choice functions can be completely characterized by the set of augmented median voter rules. See Moulin (1980),

Definition 6 (Worst-restriction). *A profile* ρ *is* worst-restricted *if, for every triple of alternatives,* $(x, y, z) \in X^3$, *there exists an* $a \in \{x, y, z\}$ *such that for each* $i \in N$, $a P_i b_i$ *for some* $b_i \in \{x, y, z\} \setminus \{a\}$.

In words, a profile is worst-restricted if for every triple $(x, y, z) \in X^3$, there is some element of that triple that no individual ranks last relative to the other two elements of the triple.

Definition 7 (α-Restriction). *A preference profile* ρ *is* α-restricted *if there do not exist two agents,* $i, j \in N$, *and four alternatives* w, x, y, *and* z *such that*

1. *The preferences over* w, x, *and* z *are opposite:* $w P_i x P_i z$ *and* $z P_j x P_j w$.
2. *The players both rank* y *higher than* x: $y P_i x$ *and* $y P_j x$.

Theorem 1 (Ballester and Haeringer 2011). *A preference profile* ρ *is* single-peaked—ρ $\in \mathcal{S}$—*if and only if it satisfies worst-restriction and* α-*restriction.*

Ubeda (2003) has recently used a different domain restriction, the two-free triple domain $(\mathcal{T}_2^n)$,[13] to demonstrate that on any domain satisfying this restriction, any weakly Paretian and IIA aggregation rule must be neutral, where neutrality characterizes those aggregation rules that disregard the labeling of alternatives when making pairwise comparisons. Because our article is concerned with collective choice rules and not preference aggregation rules per se, we omit the technical details of Ubeda's result and our previously published Theorem 2 below. The key distinction between Ubeda's result and ours is that the two-free triple domain and the single-peaked domain are not nested. In other words, satisfaction of either the two-free triple restriction or single-peakedness does not imply satisfaction of the other. We provide Theorem 2 below because it demonstrates the implications of Arrow's axioms on the single-peaked domain and thus is substantively linked to our results that follow concerning manipulation on this domain. In the third section, we will more explicitly discuss how the implications of single-peakedness for Arrow's result differ from the implications for the Gibbard-Satterthwaite theorem.

Border and Jordan (1983) and Austen-Smith and Banks (2004, 41, Theorem 2.4).

[13]This domain restriction says that for any triple of alternatives, only two orderings of the triple are possible across all individuals. While profiles on this domain will satisfy worst-restriction, they may fail α-restriction, with a clear example being the case with two individuals with preferences: $w P_1 y P_1 x P_1 z$ and $z P_2 y P_2 x P_2 w$. Similarly, the following three-player profile is single-peaked but is not an element of the two-free triple domain: $x P_1 y P_1 z$, $y P_2 x P_2 z$, and $z P_3 x P_3 y$.

Theorem 2 (Gailmard, Patty, and Penn 2008). *Let F be a weakly Paretian and IIA preference aggregation rule. If* $\mathcal{D} = \mathcal{S}$, *then F is neutral.*

Manipulation on Single-Peaked Domains

In this section, with the primitives from the previous section in hand, we state and prove our main results. First we prove that single-peakedness is not sufficient to ensure that groups of individuals cannot be made better off by submitting insincere ballots; in other words, any nondictatorial collective choice function is necessarily manipulable by a coalition at a single-peaked profile of preferences. Then we show that a collective choice function being manipulable by a coalition implies that it cannot be a dominant strategy equilibrium for all individuals to truthfully reveal their preferences. *Thus, when a nondictatorial institution is required to consider all possible collections of ballots as potential inputs, truth telling can never be a dominant strategy for all individuals, even when preferences are single-peaked.*

The Impossibility of Coalitional Strategy-Proofness

To prove that coalitional strategy-proofness implies dictatorship on preference domain $\mathcal{D} = \mathcal{S}$, we utilize the following lemmas and theorem.

Lemma 1. *Let* ϕ *be a coalitionally strategy-proof collective choice function with full range. If* $\mathcal{D} = \mathcal{S}$, *then* ϕ *is weakly Paretian.*

Proof: Consider a ρ $\in \mathcal{S}$ such that for all $i \in N$, $x P_i y$, but ϕ(ρ) $= y$. By full range, $\exists \rho' \in \mathcal{P}^n$ with ϕ(ρ') $= x$. Then ϕ(ρ')P_iϕ(ρ) for all $i \in N$, and ϕ is manipulable by coalition N. It follows that ϕ(ρ) $\neq y$ if ϕ is CSP. □

Lemma 2. *Let* ϕ *be a coalitionally strategy-proof collective choice function with full range. If* $\mathcal{D} = \mathcal{S}$, *then* ϕ *is monotonic.*

Proof: Suppose that ρ and ρ' are single-peaked, but violate monotonicity, with ϕ(ρ) $= x$, ϕ(ρ') $= y$, and $P(x, y; \rho) \subseteq P(x, y; \rho')$. We will show that this implies ϕ is not CSP on $\mathcal{S}$. Throughout the proof, let $P(x, y; \rho) = A$ and $P(y, x; \rho') = B$. We know that $A \cap B = \emptyset$.

First, to simplify notation, change ρ so that for all $i \in A$, P_i is replaced by a new preference ordering P_A. We construct P_A so that x is top ranked and y is as high in

the P_A ranking as possible while maintaining P_A single-peaked with respect to Q_ρ. It is easy to verify that such an ordering exists.

Similarly, change ρ' so that for all $j \in B$, P'_j is replaced by a new preference ordering P'_B that is constructed so that y is top ranked and x is as high as possible in the P'_B ordering while maintaining that P'_B is single-peaked with respect to $Q_{\rho'}$. CSP implies that for each of these new profiles (which in an abuse of notation we will still call ρ and ρ'), $\phi(\rho) = x$ and $\phi(\rho') = y$.

Consider the triple, x, y, $z \in X$, with z arbitrary. ρ, $\rho' \in \mathcal{S}$ imply that for each profile, one of these elements cannot be lowest ranked by any individual. If at profile ρ the two remaining elements are a, b, we say that (a, b) is lowest ranked for $\rho|_{\{a,b,z\}}$.

Construct a new profile $\rho^* \in \mathcal{P}^n$ in which $P_i^* = P_A$ for $i \in A$, and $P_j^* = P'_B$ for $j \in B$. Note that rankings are unspecified for $k \notin A \cup B$, if $N \setminus \{A \cup B\} \neq \emptyset$. These individuals' rankings can be arbitrarily assigned. Also note that ρ^* may not be single-peaked; our choice function is still required to produce an outcome at this profile, however. Then it must be the case that $\phi(\rho^*) \notin \{x, y\}$, else either coalition $N \setminus A$ could manipulate ρ with ρ^*, or $N \setminus B$ could manipulate ρ' with ρ^*. Thus, $\phi(\rho^*) = z$.

Note that in constructing ρ (and ρ^*) above, we have ensured that the only instances in which $z P_A y$ are those in which z lies between x and y under the ordering Q_ρ (i.e., $x Q_\rho z Q_\rho y$, or the reverse). This is because, while maintaining both single-peakedness and x as top ranked for P_A, y is ranked as high as possible for P_A. Similarly, $z P'_B x$ implies that z lies between x and y according to $Q_{\rho'}$.

Case 1: _(x, y) is lowest ranked for either $\rho|_{\{x,y,z\}}$ or for $\rho'|_{\{x,y,z\}}$. Without loss of generality, assume that (x, y) is lowest ranked for $\rho|_{\{x,y,z\}}$. This implies that for all individuals $k \notin A$, $y P_k x \Rightarrow z P_k x$. Thus, ϕ is manipulable at ρ by coalition $N \setminus A$ submitting ballots as in ρ^*; these individuals can guarantee themselves the outcome z, which they all prefer to x. It follows that ϕ is not CSP._

Case 2: _z is a lowest ranked element of both $\rho|_{\{x,y,z\}}$ and $\rho'|_{\{x,y,z\}}$. Note that by the construction of ρ and ρ' above, $y P_A z$ and $x P'_B z$._[14]

To recap, we have that all members of A (resp. B) have the same preference ordering over all alternatives and that this ordering ranks $x P_A y P_A z$ (resp. $y P'_B x P'_B z$). Note also that x, y, z need not appear consecutively

in these individuals' rankings. Furthermore, ρ^* as constructed above still yields z as its choice. However, now members of coalition $N \setminus A$ (resp. $N \setminus B$) may rank z last relative to x and y and may not have an incentive to manipulate ϕ by submitting ballots as in ρ^*.

We will now construct a new $\rho^o \in \mathcal{S}$ and show that ϕ CSP for ρ implies that ϕ not CSP for ρ^o, thus establishing a contradiction. This will take several intermediate steps. First, consider the ordering over alternatives induced by the preferences of individuals in A under ρ; we will call this ordering Q_A, and we know that under this ordering $x Q_A y Q_A z$. Construct a new profile $\hat{\rho}$ where $\hat{P}_i = P_A$ for all $i \in A$. Clearly this $\hat{P}_i$ is single-peaked with respect to the ordering Q_A, as it _is_ this ordering. For all $j \notin A$, assign each member of j an identical preference ordering $\hat{P}_j$ that ranks y at the top of the ballot, ranks $z \hat{P}_j x$, and is single-peaked with respect to Q_A. Such an ordering exists because y lies between z and x according to Q_A. Thus, $\hat{\rho} \in \mathcal{S}$.

We know that coalition $N \setminus A$ can submit ballots as in ρ^* and receive z as an outcome, which they prefer to x. Thus, $\phi(\hat{\rho}) \neq x$. Furthermore, $\phi(\hat{\rho}) \neq z$, by weak Pareto, because for all $i \in N$, $y \hat{P}_i z$. And by CSP of ρ, $\phi(\hat{\rho}) \neq y$, because then ρ would be manipulable by coalition $N \setminus A$ submitting ballots as in $\hat{\rho}$. Thus, if $|X| = 3$, the proof is complete. If not, it follows that $\phi(\hat{\rho}) = w$, with all members $j \in N \setminus A$ ranking $y \hat{P}_j w \hat{P}_j z \hat{P}_j x$. However, this implies that for all $i \in A$, $w P_A y$, otherwise $\hat{\rho}$ would be manipulable by coalition A submitting ballots identical to those for $N \setminus A$, and ensuring an outcome of y by weak Pareto. Thus, Q_A ranks the alternatives $x Q_A w Q_A y Q_A z$.

Now consider a new profile $\hat{\rho}^1$ in which everyone in A ranks the alternatives as in $\hat{\rho}$. For all $j \in N \setminus A$, assign each j an identical preference ordering that ranks y at the top, ranks $z \hat{P}_j^1 w \hat{P}_j^1 x$, and is single-peaked with respect to Q_A. Again, such an ordering exists given that Q_A ranks the alternatives $x Q_A w Q_A y Q_A z$. Thus, $\hat{\rho}^1 \in \mathcal{S}$. Using the same logic as above, we get that $\phi(\hat{\rho}^1) \neq x$ or w (else $N \setminus A$ manipulates with ρ^* to get z), that $\phi(\hat{\rho}^1) \neq z$ by Pareto efficiency, and that $\phi(\hat{\rho}^1) \neq y$ by CSP of our original ρ. If $|X| = 4$, the proof is complete. Otherwise, $\phi(\hat{\rho}^1) = w^1$, with individuals $j \in N \setminus A$ ranking $y \hat{P}_j^1 w^1 \hat{P}_j^1 z \hat{P}_j^1 w \hat{P}_j^1 x$. Again, it follows that for all $i \in A$, $\hat{P}_i^1 = P_A$ ranks $w^1 \hat{P}_i^1 y$; otherwise this coalition would manipulate $\hat{\rho}^1$ in order to get y as the outcome. Thus, Q_A ranks the alternatives $x Q_A \{w^1, w\} Q_A y Q_A z$ (the w^1, w ranking need not be specified in the proof, although by coalition $N \setminus A$'s preferences, the ranking must respect $w Q_A w^1$).

Repeat the above steps for $k = 2, \ldots, |X| - 4$ by choosing a new $\hat{\rho}^k$ with $y \hat{P}_j^k z \hat{P}_j^k \ldots$ for all $j \in N \setminus A$, and $\hat{P}_i^k = P_A$ for $i \in A$. Such a $\hat{\rho}^k$ can be constructed so as to always remain single-peaked with respect to Q_A

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[14]This immediately implies that ϕ does not satisfy Pareto efficiency on the full domain $\mathcal{P}^n$, because ρ^* can be constructed so as to have all individuals not in $A \cup B$ place x and y at the top of their ballots, and so $x P_k^* z$ and $y P_k^* z$ for all k, and yet $\phi(\rho^*) = z$.

because it must always be the case that the social choice at each stage, w^k, is ranked between x and y for all $i \in A$. We ultimately get that Q_A ranks the alternatives $x\,Q_A\,\{w, w^1, w^2, \ldots w^{K-4}\}\,Q_A\,y\,Q_A\,z$. At this point, construct ρ^o so that P_j^o is such that $y\,P_j^o\,z\,P_j^o\,\{w, w^1, \ldots, w^{|X|-4}\}\,P_j^o\,x$ for all $j \in N\backslash A$ and $P_i^o = P_A$. Again, ρ^o is single-peaked with respect to Q_A. CSP requires $\phi(\rho^o)\,P_j^o\,z$ for all $j \in N\backslash A$; otherwise this coalition would manipulate with ballots as in ρ^*. CSP of ϕ at ρ requires $\phi(\rho^o) \neq y$. And these two statements imply a contradiction. Thus, ϕ is not CSP. It follows that ϕ CSP implies ϕ monotonic when $\mathcal{D} = \mathcal{S}$. □

For the next lemma, we will need the following two definitions:

Definition 8 (Blocking coalition for (x, y)). *A coalition $L \subseteq N$ is a blocking coalition for (x, y) if for all $\rho = (P_1, \ldots, P_n) \in \mathcal{D}$ such that $x\,P_i\,y$ for all $i \in L$ and $y\,P_j\,x$ for all $j \notin L$, $\phi(\rho) \neq y$.*

Definition 9 (Blocking coalition). *A coalition $L \subseteq N$ is a blocking coalition if for all $\rho = (P_1, \ldots, P_n) \in \mathcal{D}$ and all pairs $(a, b) \in X^2$, $a\,P_i\,b$ for all $i \in L \Rightarrow \phi(\rho) \neq b$.*

Lemma 3. ϕ *coalitionally strategy-proof with full range implies that if there exists one $\rho \in \mathcal{S}$ with $x\,P_i\,y$ for all $i \in L$ and $y\,P_j\,x$ for all $j \notin L$ and $\phi(\rho) = x$, then L is a blocking coalition.*

Proof: Let ρ be such that $x\,P_i\,y$ for all $i \in L$ and $y\,P_j\,x$ for all $j \notin L$ and $\phi(\rho) = x$. We will first show that L is blocking for (x, y), and then that L is a blocking coalition.

Suppose that L is not blocking for (x, y), so that there exists a $\rho' \in \mathcal{S}$ with $x\,P_i'\,y$ for all $i \in L$ and $y\,P_j'\,x$ for all $j \notin L$ and $\phi(\rho') = y$. By monotonicity, this implies that $\phi(\rho) \neq x$, a contradiction. Thus, L is blocking for (x, y).

We will now show that L blocking for (x, y) implies that for any $a \notin \{x, y\}$, L is blocking for (x, a) and for (a, y). Consider any $\rho \in \mathcal{S}$ where $i \in L$ rank the alternatives $x\,P_i\,y\,P_i\,a$, $j \notin L$ rank them $y\,P_j\,a\,P_j\,x$, and all $k \in N$ have $c\,P_k\,d$ when $c \in \{a, x, y\}$ and $d \notin \{a, x, y\}$. Then $\phi(\rho) = x$, by L blocking for (x, y) and by weak Pareto. Thus, L is blocking for (x, a).

Now construct a $\rho' \in \mathcal{S}$ with $P_j' = P_j$ for $j \notin L$ and with $i \in L$ having $a\,P_i\,x\,P_i\,y$ and $c\,P_i\,d$ when $c \in \{a, x, y\}$ and $d \notin \{a, x, y\}$. In this case $\phi(\rho') = a$, again by L blocking for (x, y) and by weak Pareto. Thus, L is blocking for (a, y).

Because a was chosen at random, the above argument proves that L is also blocking for any distinct pair (c, d): L blocking for (x, y) implies L blocking for (c, y), and this implies L blocking for (c, d), for any $d \neq y$.

Last, by monotonicity, we will show that L blocking for all (a, b) implies that at any profile $\rho \in \mathcal{S}$ in which $a\,P_i\,b$ for all $i \in L$, then $\phi(\rho) \neq b$. Suppose not; suppose

that $\phi(\rho) = b$. Let Q be the ordering that ρ is single-peaked with respect to. Now consider a ρ' that is also single-peaked with respect to Q, in which $P_i' = P_I$ for all $i \in L$, and for each $j \notin L$, P_j is replaced by a P_j' in which b is top ranked for P_j'. By monotonicity, $\phi(\rho') = b$. However, under ρ' we have $a\,P_i\,b$ for all $i \in L$ and $b\,P_j\,a$ for all $j \notin L$. $\phi(\rho') = b$ contradicts L blocking for (a, b). Thus, L is a blocking coalition. □

Theorem 3. *Let ϕ be a coalitionally strategy-proof collective choice function with full range. If $\mathcal{D} = \mathcal{S}$, then ϕ is dictatorial.*

Proof: Consider three profiles $\rho_1, \rho_2, \rho_3 \in \mathcal{S}$ in which the alternatives x, y, z are at the top of each person's preference ordering, and all other alternatives are ordered according to a fixed ordering $Q_{-\{xyz\}}$. Thus, save for alternatives $\{x, y, z\}$, rankings over all other alternatives are identical across all individuals and all three profiles.

Let $L \subseteq N$ be a "minimal" blocking coalition. Thus, for any $i \in L$, the set $L\backslash\{i\}$ is not a blocking coalition. Such a coalition exists because, by Pareto, we know that the collection of blocking coalitions is nonempty. Define $\{x, y, z\}$ rankings under ρ_1, ρ_2, ρ_3 as follows:

	ρ_1	ρ_2	ρ_3
i	$x\,P\,z\,P\,y$	$x\,P\,y\,P\,z$	$x\,P\,y\,P\,z$
$L\backslash\{i\}$	$y\,P\,z\,P\,x$	$y\,P\,x\,P\,z$	$y\,P\,z\,P\,x$
$N\backslash L$	$z\,P\,x\,P\,y$	$z\,P\,x\,P\,y$	$z\,P\,y\,P\,x$

We know the following: $\phi(\rho_2) \neq z$ because all in L prefer y to z; $\phi(\rho_3) \neq z$ because all in L prefer y to z; $\phi(\rho_1) \neq y$ because everyone *not* in $L\backslash\{i\}$ prefers z to y, and we have assumed that $L\backslash\{i\}$ is not a blocking coalition; $\phi(\rho_2) \neq y$ because everyone *not* in $L\backslash\{i\}$ prefers x to y, and we have assumed that $L\backslash\{i\}$ is not a blocking coalition. These last two statements are consequences of Lemma 3.

Condensing the above paragraph, we now know that $\phi(\rho_1) = x$ or z, that $\phi(\rho_2) = x$, and that $\phi(\rho_3) = x$ or y.

Case 1: *First, suppose that $\phi(\rho_1) = z$. This implies that $\phi(\rho_3) = y$, because (x, z) preferences are identical across ρ_1 and ρ_3. Thus, $\phi(\rho_3) = x$ would violate monotonicity.*

Now consider an insincere ballot $\hat{\rho}$ that is identical to ρ_1, ρ_2, ρ_3 for all $w \notin \{x, y, z\}$ (i.e., these alternatives are, for every individual, ordered according to $Q_{-\{xyz\}}$), and with a Condorcet cycle over x, y, z at the top:

We know that $\phi(\hat{\rho}) = d \notin \{x, y, z\}$; otherwise ρ_1, ρ_2, or ρ_3 would be manipulable by an individual or a coalition submitting ballots as in $\hat{\rho}$: $\phi(\hat{\rho}) = x \Rightarrow i$

	$\hat{\rho}$
i	$x P y P z$
$L \setminus \{i\}$	$y P z P x$
$N \setminus L$	$z P x P y$

manipulates ρ_1, $\phi(\hat{\rho}) = y \Rightarrow L \setminus \{i\}$ manipulates ρ_2, and $\phi(\hat{\rho}) = z \Rightarrow N \setminus L$ manipulates ρ_3.

Now, construct a new profile $\hat{\rho}_1 \in \mathcal{S}$ with the preferences of all $j \neq i$ identical to those given by $\hat{\rho}$. Player i's new preferences rank d highest, with i's rankings over all $w \notin \{d, x, y, z\}$ unchanged. This profile is single-peaked according to the ordering specified by i's preferences, which can be verified by considering α- and worst-restriction. Since all players have identical orderings over alternatives not in $\{d, x, y, z\}$, moving d to the top of i's ranking cannot break worst-restriction (as all other players have the same ranking of d and any other alternative $a \neq d$), and cannot break α-restriction, as only player i has preferences over a triple that are the reverse of another player's preferences, and i is the unique player with d at the top of his ballot.

$\phi(\hat{\rho}_1) = d$, otherwise $\phi(\hat{\rho}_1)$ would be manipulable by i submitting a ballot as in $\hat{\rho}$. But, by Lemma 3, this implies that i is a blocking coalition, because all other players prefer x, y, and z to d. Thus, i is a dictator on $\mathcal{S}$.

Case 2: *It must now be the case that $\phi(\rho_1) = x$. Then immediately, by Lemma 3, this implies that i is a blocking coalition, because i is the unique person who prefers x to z at profile ρ_1. Thus, i is a dictator on $\mathcal{S}$.*

We have shown that i is a blocking coalition on preference domain $\mathcal{S}$. To finish, we must show that i is a dictator on the full ballot domain. Suppose not, so that there exists $\rho \in \mathcal{P}^n$ with $\phi(\rho) = y$ and with $y \neq x_i^*(\rho)$, with $x_i^*(\rho)$ being i's reported ideal point under ballot profile ρ. We also know that there exists a $\rho' \in \mathcal{S}$ with $P_i = P_i'$ and with $y = x_j^*(\rho')$ for all $j \neq i$.[15] Voter i being a blocking coalition on $\mathcal{S}$ implies $\phi(\rho') = x$. However, this contradicts ϕ coalitionally strategy-proof, because $\phi(\rho')$ would be manipulable by $N \setminus \{i\}$ submitting ballots as in ρ and attaining their ideal point, y. It follows that i is a dictator on the full ballot domain. $\square$

When $n = 3$, Theorem 3 can be strengthened to say that if ϕ is a *strategy-proof* collective choice function and $\mathcal{D} = \mathcal{S}$, then ϕ is dictatorial. However, we cannot weaken coalitional strategy-proofness to strategy-proofness when

[15]Such a $\rho' \in \mathcal{S}$ would exist and be single-peaked, for example, for an ordering of the policy space Q equal to i's preference ordering, so that xQy if and only if $x P_i y$.

$n \geq 5$. This point, along with what strategy-proofness and coalitional strategy-proofness imply about dominant strategy and Nash implementation, is explored in the next section.

Strategy-Proofness and Truth Telling as a Dominant Strategy Equilibrium

It is important to note that on a restricted preference domain and unrestricted ballot domain, strategy-proofness of a choice function is no longer equivalent to that choice function being truthfully implementable in dominant strategies by a direct mechanism. Strategy-proofness simply tells us that at any *truthful* profile of ballots, no individual can submit an insincere ballot and strictly benefit from that insincerity. It says nothing about whether individuals may benefit from insincerity at profiles of ballots in which other individuals are also being insincere. When the ballot domain and preference domain are equivalent, the definition of strategy-proofness implies dominant strategy implementability, because every possible strategy (or ballot) profile is also a preference profile. However, in the setting we consider, the unrestricted domain of ballot profiles (or strategy profiles) is larger than the collection of single-peaked preference profiles.

In the next theorem, we will leverage our previous result on coalitional strategy-proofness to show that, when considering an unrestricted ballot domain, there exists no nondictatorial choice function that is implementable in dominant strategies when true preferences are single-peaked. When preferences are single-peaked, there do, however, exist strategy-proof collective choice functions, or choice functions truthfully implementable in Nash equilibrium. These results highlight the fact that on the single-peaked domain we consider, the typical equivalence between coalitional strategy-proofness and strategy-proofness does *not* hold. The only coalitionally strategy-proof choice function on $\mathcal{S}$ is dictatorial. However, in Theorem 5, we provide an example of a nondictatorial choice function that is strategy-proof on $\mathcal{S}$. It is not, however, dominant strategy implementable. Prior to presenting and discussing these results, however, we must introduce some notation.

For any individual $i \in N$, $\mathcal{M}_i = \mathcal{P}$ is the collection of actions, or *messages*, available to individual i, and $\mathcal{M} \equiv \times_{i \in N} \mathcal{M}_i$ is the set of all profiles of messages. Recall that $\mathcal{P}$ is the collection of all strict orders on X. Thus, an individual's message is an ordering of the policy space. An *outcome function* g maps a profile of messages $m = (m_1, \ldots, m_n) \in \mathcal{M}$ into the policy space, X. The pair $(\mathcal{M}, g)$ is a *mechanism*, or game form. If the collection of

messages available to each individual is equal to the set of preferences that the individual may have, the mechanism is called *direct*.

Given a mechanism $(\mathcal{M}, g)$, a *strategy* for person i is a function σ_i mapping preference profiles $\rho \in \mathcal{D}$ into messages, so that $\sigma_i(\rho) = m_i$. A *game*, G, consists of a mechanism and a preference profile, so that $G = ((\mathcal{M}, g), \rho)$. For any game G, a strategy profile $\sigma^*(\rho) \in \mathcal{M}$ is

- a *dominant strategy equilibrium for G* if and only if for all $i \in N$, all messages $m_i \in \mathcal{M}_i$ and all $m_{-i} \in \times_{j \neq i} \mathcal{M}_j$,

$$g(\sigma_i^*(\rho), m_{-i}) R_i g(m_i, m_{-i}),$$

- a *Nash equilibrium for G* if and only if for all $i \in N$ and all $m_i \in \mathcal{M}_i$,

$$g(\sigma_i^*(\rho), \sigma_{-i}^*(\rho)) R_i g(m_i, \sigma_{-i}^*(\rho)).$$

In words, then, a dominant strategy equilibrium is one in which the best choice for any individual i, $\sigma_i^*(\rho)$, is independent of the strategies played by other players: it is an unconditionally optimal choice. In a Nash equilibrium, on the other hand, the strategy of any given player is optimal, *given the Nash equilibrium strategies being played by other players*.

A choice function ϕ with preference domain $\mathcal{D}$ and ballot domain $\mathcal{P}^n$ is

- *truthfully implementable in dominant strategy equilibrium* if and only if there exists a direct mechanism $(\mathcal{M}, g)$ so that for all $\rho \in \mathcal{D}, \sigma^*(\rho) = \rho$ is a dominant strategy equilibrium for the game $G = ((\mathcal{M}, g), \rho)$, and $g(\rho) = \phi(\rho)$,
- *truthfully implementable in Nash equilibrium* if and only if there exists a direct mechanism $(\mathcal{M}, g)$ so that for all $\rho \in \mathcal{D}, \sigma^*(\rho) = \rho$ is a Nash equilibrium for the game $G = ((\mathcal{M}, g), \rho)$, and $g(\rho) = \phi(\rho)$.

With these definitions in hand, the next theorem formally links our social choice result, Theorem 3, with implementation theory.

Theorem 4. *If ϕ is coalitionally manipulable, then ϕ is not truthfully implementable in dominant strategy equilibrium.*

Proof: Let ϕ be coalitionally manipulable. Then there exists a coalition $L \subseteq N$, a $\rho \in \mathcal{D}$, and a $\rho' \in \mathcal{P}^n$ such that $\phi(\rho'_L, \rho_{-L}) P_i \phi(\rho)$ for all $i \in L$. Number the individuals in N so that the individuals in L are numbered from 1 through $|L|$. Coalitional manipulability of ϕ implies that there exists an $i \in L$ such that $\phi(\rho'_{\{1, \dots, i-1\}}, P'_i, \rho_{N \setminus \{1, \dots, i\}}) P_i \phi(\rho'_{\{1, \dots, i-1\}}, P_i, \rho_{N \setminus \{1, \dots, i\}})$. Thus, $\sigma_i(\rho) = P_i$ is not a dominant strategy for person i. It follows that

ϕ is not truthfully implementable in dominant strategy equilibrium.   □

The following Corollary, which follows immediately from Theorems 3 and 4, summarizes an important implication of our results that we return to in a following section. Specifically, the corollary implies that designing deliberative institutions so as to implement the majority will require accounting for obfuscation so long as individuals are strategic: *sincere revelation cannot be made a weakly dominant strategy*.

Corollary 1. *When $\mathcal{D} = \mathcal{S}$, there exists no nondictatorial collective choice function that is truthfully implementable in dominant strategies. In particular, a collective choice function that always returns a Condorcet winner when one exists is not truthfully implementable in dominant strategies.*

While Theorem 4 proved that coalitional manipulability precludes the possibility of truthful dominant strategy implementation, it does not rule out the possibility of truthful Nash implementation. However, as noted earlier, when $n = 3$, Theorem 3 can be strengthened to say that if ϕ is a *strategy-proof* collective choice function and $\mathcal{D} = \mathcal{S}$, then ϕ is dictatorial. Thus, when $n = 3$ and $\mathcal{D} = \mathcal{S}$, no nondictatorial choice function is truthfully implementable in Nash equilibrium.

We cannot weaken coalitional strategy-proofness to strategy-proofness when $n \geq 5$, and therefore, in this situation we can design a mechanism in which the truthful revelation of preferences is a Nash equilibrium. This is because with four or more voters and a single person submitting an insincere ballot, the set of potential manipulators can be narrowed to two or fewer individuals in any situation in which the submitted profile of ballots is not single-peaked.[16] In the absence of a core alternative in the submitted profile of ballots, removing these individuals would yield a profile admitting a nonempty core, and even if the resulting core is multivalued, the individual can never profit from having his ballot dropped.[17]

Theorem 5. *For any $n \geq 5$, there exists a nondictatorial choice function possessing full range that is*

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[16]This is to say that we may not be able to uniquely identify an insincere ballot, but we can identify a unique pair of individuals, one of whom has submitted an insincere ballot.

[17]Defining the institution's behavior in instances in which the core of a sincere ballot profile is potentially multivalued requires some care. In particular, without knowledge of the underlying ordering of alternatives, Q, we cannot break ties within the core in a way that is not manipulable by an individual. However, in this case we can specify an individual whose ballot is always discarded, thus bringing us back to an effectively odd number of players. When $n = 4$, however, this technique does not work because it reduces us to a three-player world.

- *strategy-proof on the single-peaked preference domain and unrestricted ballot domain and*
- *truthfully implementable in Nash equilibrium.*

Proof. Fix $n \geq 5$ and let Q represent any ordering of the k alternatives in X, with Q ordering the k alternatives $x_1, \ldots, x_k$. Let ρ_{-L} be the ballot profile generated by dropping the ballots of all individuals in $L \subset N$ and retaining all other ballots. Let $C(\rho) \in 2^X$ be the collection of majority core alternatives at profile ρ, so that $C(\rho) = \{x \in X : \text{for all } y \in X, |i : x P_i y| \geq \frac{n}{2}\}$. Consider the following "Drop 2" choice function on $\rho \in \mathcal{P}^n$:

1. If n is odd and ρ satisfies both worst-restriction and α-restriction, then $\phi(\rho) = C(\rho)$.
2. If n is even and ρ satisfies both worst-restriction and α-restriction, then $\phi(\rho) = C(\rho_{-\{1\}})$.
3. If ρ violates one of these conditions, then for each $i = 1, \ldots, n$ test whether $\rho_{-\{i\}}$ satisfies both worst-restriction and α-restriction. If so, let $i \in W$.
   (a) If $|W| < n$ and n is odd, then $\phi(\rho) = \operatorname{argmin}_{x_j \in C(\rho_{-W})}\{j\}$.
   (b) If $|W| < n$ and n is even, then $\phi(\rho) = \operatorname{argmin}_{x_j \in C(\rho_{-\{W \cup \{1\}\}})}\{j\}$.
   (c) If $|W| = n$, then $\phi(\rho) = x_1$.

We will consider two cases: the case in which a ballot profile is single-peaked, and the case in which it is not. Let $\rho^* = (P_1, \ldots, P_n) \in \mathcal{S}$ be a sincere preference profile, with $\phi(\rho^*) = x^*$, and throughout, let $\succ_\rho$ denote the strict majority preference relation induced by a profile ρ.

First, suppose that i can profitably manipulate ϕ at profile ρ^* with ballot P_i', and that (P_i', ρ_{-i}^*) is single-peaked. Assuming that n is odd, this implies that $x = C(P_i', \rho_{-i}^*) \neq C(\rho^*)$, and that $x P_i x^*$. However, we know that $x^* \succ_{\rho^*} x$, and, because $x P_i x^*$, that $x^* \succ_{(P_i', \rho_{-i}^*)} x$, contradicting the fact that x is the core of (P_i', ρ_{-i}^*). If n is even, this same logic holds, with p^* replaced by $p_{-\{1\}}^*$.

Second, suppose that (P_i', ρ_{-i}^*) is not single-peaked. Since i is the sole person submitting an insincere ballot, it follows that $i \in W$. It may also be the case that there exists one other $j \in W$; however, if $\rho^* \in \mathcal{S}$, then $|W| \leq 2$.[18] First, suppose n is odd. Then for this manipulation to be profitable for i, we know that $x = \phi(P_i', \rho_{-i}^*) \in C(\rho_{-W})$, and that $x P_i x^*$. Let $W = \{i\}$ or $W = \{i, j\}$. Because $x^* \succ_{\rho^*} x$, then $x^* \succ_{\rho_{-W}} x$ also, and regardless of whether

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[18]This is because with $n \geq 5$, at least two people will rank any element of a triple last; thus, worst-restriction could be violated by at most two ballots. Furthermore, α-restriction can only be violated by a pair of individuals, and the true manipulator will be in any such pair. Thus, α-restriction can be violated by at most two individuals.

W contains one or two individuals. Either one or two supporters of x are removed and so x^* is still majority-preferred to x, or a supporter of x and a supporter of x^* are removed, thus canceling each other out. In either case, $x^* \succ_{\rho_{-W}} x$, contradicting $x \in C(\rho_{-W})$. Supposing now that n is even, the same argument holds, replacing ρ^* with $\rho_{-\{1\}}^*$ as appropriate.

Finally, having shown that this choice function is strategy-proof is equivalent to having shown that everyone submitting a sincere ballot, or $\sigma_i^* = P_i$ for all i, is a Nash equilibrium of the game $((\mathcal{P}^n, \phi), \rho)$, for all $\rho \in \mathcal{S}$. Thus, ϕ is truthfully implementable in Nash equilibrium. $\square$

Examples of Manipulation in One Dimension

In this section, we consider two applications that illustrate the broader implications of our findings. The first example, discussed next, demonstrates a feature of our results that may not be apparent from the theoretical sections of this article, and that is important to note. As discussed in the introduction, domain restrictions pose different challenges when considering Arrow's theorem versus when one considers the Gibbard-Satterthwaite theorem. While coalitional strategy-proofness implies dictatorship of choice functions on $\mathcal{S}$, IIA and weak Pareto only imply neutrality of preference aggregation rules on $\mathcal{S}$.
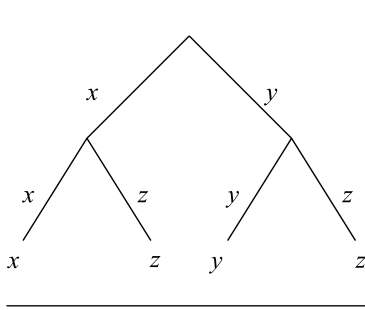
We first offer an explicit illustration of this distinction between Arrow and Gibbard-Satterthwaite in the domain of single-peaked preferences by considering a canonical example of a generally nonneutral institution—the amendment voting tree—that happens to be neutral on $\mathcal{S}$, that satisfies IIA and weak Pareto on $\mathcal{S}$, and that is also manipulable on $\mathcal{S}$. Following that, we present a simplification of the argument we use in our main theorem, Theorem 3. It demonstrates that the incentive to manipulate will manifest itself as an individual or a coalition either attempting to make the policy space appear multidimensional or attempting to manipulate the true dimension of conflict. We then discuss the implications of the argument for recent work on deliberative democracy.

Amendment Agendas

In this section, we briefly consider an institution that has received much attention in formal models of politics: the amendment agenda. Under an amendment agenda, alternatives are voted upon in an ordered sequence of pairwise votes. It is well known that in the absence of a Condorcet

FIGURE 1   A Two-Stage Amendment Agenda

winner, these agenda procedures are highly manipulable, with any alternative in the top cycle being attainable as a policy outcome depending on the sequence of votes taken. Thus, amendment agendas are, in general, not neutral, because they privilege alternatives appearing later on in the agenda.[19] It is also well known that in the presence of a Condorcet winner, any sequence of voting will yield the Condorcet winner as an outcome, regardless of whether all individuals vote sincerely or all vote sophisticatedly.[20] Thus, over the domain $\mathcal{S}$, amendment agendas *are* neutral: if the collection of ballots an amendment agenda is given is single-peaked, then so is any relabeling of that collection, and the outcome of voting will be the Condorcet winner (and the relabeled Condorcet winner) of each ballot profile.

What is perhaps less well known is that amendment agendas are highly manipulable at sincere profiles of ballots, even in the presence of a Condorcet winner.[21] In this section, we consider amendment agendas to be choice functions, ϕ_A, in which an individual's ballot dictates how that individual will vote on any pair of alternatives.
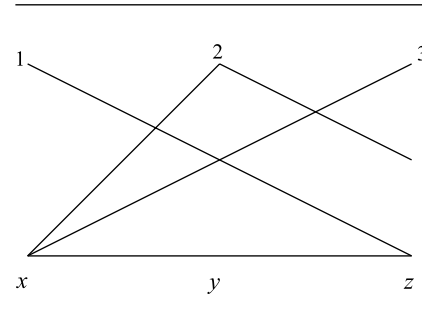
In the presence of a Condorcet winner, assuming that all individuals vote either sincerely or sophisticatedly yields the same outcome. However, the sequence of votes that individuals cast will differ, and at a sincere profile of ballots, or sequence of votes, an amendment agenda is manipulable. To see this, consider the amendment agenda pictured in Figure 1, in which alternatives x and y are first put to a vote via majority rule, and the

winner is then pitted against z in order to determine the final outcome. Suppose that there are three individuals with the following preferences: $x P_1 y P_1 z$, $y P_2 z P_2 x$, and $z P_3 y P_3 x$. This preference profile is single-peaked and is pictured graphically in Figure 2; it yields y as a Condorcet winner.

Under a truthful collection of ballots, the amendment agenda pictured in Figure 1 yields y as an outcome: y defeats x at the first stage of voting by the votes of Players 2 and 3, and y defeats z at the second stage by Players 1 and 2. Now consider a collection of ballots in which Players 1 and 2 truthfully reveal their preferences over alternatives, but Player 3 claims to have the preference ordering $z P_3' x P_3' y$. Under this ballot profile, our amendment agenda now yields z as the winner: x defeats y at Stage 1 by the (sincere) vote of Player 1 and the (insincere) vote of Player 3, and z defeats x at Stage 2 by the sincere votes of both Players 2 and 3. Furthermore, this is a beneficial manipulation by Player 3, as it enables him to attain his ideal point as the policy outcome.

Clearly the insincere ballot of Player 3 is not single-peaked with respect to the underlying ordering of alternatives. However, without a priori restricting how people can cast votes, manipulation is endemic to this form of agenda, even when the majority will is clearly well defined. And we know of no real-world institution that restricts how pairwise votes may be cast. At the same time, when handed a truthful profile of ballots, the amendment agenda produces outcomes and sequences of votes that are consistent with pairwise majority voting. Thus, contrary to the argument forwarded by Dryzek and List (2003), satisfying Arrow's conditions does not "by easy implication" imply satisfaction of the conditions of Gibbard-Satterthwaite on single-peaked domains. Pairwise majority voting is transitive, weakly Paretian and IIA (and thus, neutral) when a collection of preferences is single-peaked, and produces outcomes consistent with those produced by an amendment agenda "choice function" ϕ_A. Amendment agendas, and in general any choice function

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[19] The last alternative considered in a pairwise vote need only defeat the winning alternative that preceded it in order to become a policy outcome. However, an alternative considered first must defeat every other policy in order to be chosen as a policy outcome.

[20] Sophisticated voting in this context refers to individuals playing subgame-perfect Nash equilibrium strategies.

[21] Others have noted that when the behavioral assumption of sincerity or sophistication is not uniformly made across all voters, amendment agendas may no longer be Condorcet consistent. See Denzau, Rikev, and Shepsle (1985) and Austen-Smith (1987), among others.

FIGURE 2   A Single-Peaked Preference Profile

representing the outcome of a binary voting process, are not strategy-proof.

Deliberative Democracy

Deliberation within democratic governance has attracted a great deal of attention in the past two decades. This attention has focused primarily on the potential impact of deliberation on the quality of democratic decision making, broadly construed. For example, deliberative democracy has been forwarded as a means of divining the "best" choice (Estlund 1997; Estlund et al. 1989) as a legitimating device (e.g., Cohen 1989; Manin 1987), and as a means by which individuals' preferences may be brought more in line with each other (e.g., Miller 1992). In addition, some have argued that deliberation is in and of itself a good thing (e.g., Shapiro 2002).[22]

Dryzek and List (2003) have recently extended such arguments and explored the linkages between deliberative democracy and social choice theory. In many respects, their arguments provide hope for deliberative democrats in spite of the generally negative conclusions about the coexistence of collective rationality and democratic collective choice. The heart of their argument is that deliberative decision making may allow individuals within a group to leverage the underlying common structure of their individual preferences to choose an outcome that satisfies desirable normative (e.g., democratic) properties. The principle example of such a structure is single-peakedness. Dryzek and List link single-peakedness with *agreement at a meta-level*, a notion loosely describing the agreement by participants "on a common dimension in terms of which the alternatives are to be conceptualized" (Dryzek and List 2003, 14).

A key conclusion for Dryzek and List's purposes is that preference structuration can eliminate the incentive for a deliberator to misrepresent his or her preferences at the point at which the collective is faced with making a final decision. Claims that structuration may be produced through deliberation have been forwarded by many scholars, and some empirical evidence supports this claim.[23] Our arguments in this article provide some insight into the conditions that one must impose on the structuration process to guarantee that it offers no benefit from mis-

representation and show that these conditions are very restrictive. Indeed, the conditions are far more restrictive than has been claimed elsewhere.[24] Specifically, we have shown that a nondictatorial decision-making institution can be ensured to offer no benefit from misrepresentation by a person or coalition only if the details of the structure of the individuals' preferences are written into the rules of the institution itself (i.e., the institution can utilize the underlying ordering of alternatives). This is akin to saying that if we wish to design a deliberative institution that will give individuals the incentive to participate in a process of collective preference structuration, the institution must have a priori knowledge of the true structure of preferences; it must know the true left-right ordering of the alternative space *before* the deliberative process begins.

The following table illustrates exactly why individuals with single-peaked preferences may not want the true "dimension of conflict" revealed. It depicts three different preference profiles that are all single-peaked. Because each profile is single-peaked and there are three individuals, each profile yields a unique Condorcet winner, or policy that is the median voter's ideal point. Consider a choice function that yields a Condorcet winner whenever a Condorcet winner exists (presumably, the benefit of preference structuration is the existence of such a policy). Thus, for ρ_1 it yields z as the outcome, and so on.

What does this choice function yield when it receives a collection of ballots that does not admit a Condorcet winner? The table shows that regardless of which outcome the choice function chooses, the function is always manipulable by someone at a single-peaked profile of preferences. For example, if the choice function chooses x as the outcome when it receives a cyclic profile of ballots, then the function is manipulable by Person 1 at the (sincere) profile ρ_1. In particular, Person 1 has an incentive to report that his preferences are $x \succ y \succ z$, when they are truly $x \succ z \succ y$. By misrepresenting his preferences this way, he switches the outcome from z (the median voter's ideal point) to x (his own ideal point).

In each case, for manipulation to occur, an individual must submit a ballot that is not single-peaked with respect to the true ordering of alternatives. In this sense, if individuals are not constrained to choose only from the set of ballots that are single-peaked with respect to precisely the ordering that the deliberative process was intending to uncover, we cannot guarantee that an individual will have no incentive to lie about what he believes

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[22]Knight and Johnson (1994) present a critique of the parallels drawn (particularly as argued by Miller 1992) between preference aggregation and deliberation.

[23]Theories discussing the production of structuration through deliberation include Mansbridge (1983), Goodin (1986), and Miller (1992). Empirical studies of the emergence of structuration have been offered by Radcliff (1993) and Farrar et al. (2010).

[24]In addition to Dryzek and List (2003), our results appear to be at odds with some of the relevant claims of Grofman and Feld (1988), Miller (1992), and Mackie (2003), among others.

	ρ_1	ρ_2	ρ_3	Cycle
Person 1	$x \succ z \succ y$	$x \succ y \succ z$	$x \succ y \succ z$	$x \succ y \succ z$
Person 2	$y \succ z \succ x$	$y \succ x \succ z$	$y \succ z \succ x$	$y \succ z \succ x$
Person 3	$z \succ x \succ y$	$z \succ x \succ y$	$z \succ y \succ x$	$z \succ x \succ y$
				x: 1 manipulates ρ_1
Outcome	Median $= z$	Median $= x$	Median $= y$	**y**: 2 manipulates ρ_2
				z: 3 manipulates ρ_3

the true dimension of conflict to be, or to claim that the policy space is two-dimensional, or more complex than it truly is. In other words, even if we begin with a situation in which preferences are already structured in a desirable way, we cannot design an institution that is guaranteed to truthfully elicit those preferences.

Conclusions

Theorems 2 and 3 imply that one must be careful in interpreting collective will in any real-world policymaking institution *even when preferences are presumed to be single-peaked*. This point is highly relevant for those scholars who insist that majority rule cycles are infrequent or untroubling (e.g., Mackie 2003). Our results directly imply that—even if preferences are known to be single-peaked and accordingly admit no cycles—any nondictatorial and full-range choice institution will necessarily in some situations provide one or more groups of individuals with the ability to profit by *claiming* that preferences are cyclic. It then follows that appeals to aggregate outcomes as empirical indicators of the collective will are not necessarily well founded even when the majority will is assumed to exist. Put another way, the normative and descriptive issues raised by Arrow's theorem and the Gibbard-Satterthwaite theorem are more than "logical exercises," "abstract limit cases," or "mathematical curiosities" dreamed up for the purpose of scholarly debate, as claimed by some.[25] Single-peakedness does not solve the problems raised by Arrow's theorem in the real world because policymaking institutions are generally not neutral. Finally, and in very practical terms, single-peakedness does not eliminate the possibility of gains through strategic manipulation within real-world institutions, because few (if any) policymaking institutions are dictatorial, and few (if any) policymaking institutions limit the preferences that individuals and groups can *claim* to have.

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[25]See Mackie (2003, 156, 192).

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