Annual Review of Political Science

Measuring Fairness, Inequality, and Big Data: Social Choice Since Arrow

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Annu. Rev. Political Sci. 2019. 22:435–60

First published as a Review in Advance on February 6, 2019

The *Annual Review of Political Science* is online at polisci.annualreviews.org

https://doi.org/10.1146/annurev-polisci-022018-024704

Keywords

justice, rights, aggregation, liberalism, Pareto, independence of irrelevant alternatives

Abstract

Kenneth J. Arrow was one of the most important intellectuals of the twentieth century, and his "impossibility theorem" is arguably the starting point of modern, axiomatic social choice theory. In this review, we begin with a brief discussion of Arrow's theorem and subsequent work that extended the result. We then discuss its implications for voting and constitutional systems, including a number of seminal results—both positive and negative—that characterize what such systems can accomplish and why. We then depart from this narrow interpretation of the result to consider more varied institutional design questions such as apportionment and geographical districting. Following this, we address the theorem's implications for measurement of concepts of fundamental interest to political science such as justice and inequality. Finally, we address current work applying social choice concepts and the axiomatic method to data analysis more generally.

I believe that most of the arguments against "quantitizing" or "measuring" the "qualitative" variables encountered in the social sciences stem from ignorance of how flexible the concept "quantity" is, and how indefinite the lines between quantity and quality. Such arguments are particularly suspect when it is asserted in one sentence that a particular variable is "essentially qualitative" and in the next that the adjectives "more" or "less" can be predicated of it.

–Herbert Simon (1953, p. 513, fn.13)

1. INTRODUCTION

Kenneth J. Arrow was one of the most important intellectuals of the twentieth century. Arrow made numerous contributions to social science. However, his impossibility theorem (Arrow 1951, 1963) represents the starting point of modern social choice theory, and the axiomatic method Arrow employed in the presentation and proof of the impossibility theorem represents a cornerstone of the modern theory of measurement in social science.[1]

Specifically, Arrow laid out a set of prescriptive and normatively defensible axioms that a reasonably democratic procedure for evaluating collective preference should satisfy. He then showed that no such procedure exists; any method must violate at least one of the axioms. Arrow's formalization of collective preference and the axioms governing democratic decision making launched the modern field of social choice and set the course for its development. In the years following the 1951 publication of *Social Choice and Individual Values*, Arrow's original axioms were extended, modified, and weakened, and the field branched into new terrain as scholars applied the axiomatic approach to problems far removed from voting theory.

The work we discuss follows in Arrow's tradition by aiming to apply axiomatic and measurement-theoretic techniques to fundamental political and societal problems. While (we hope) most political scientists are familiar with the general thrust of Arrow's theorem, few have encountered much of the body of work that Arrow's theorem inspired. This is unfortunate, because—like the theorem itself—much of this work grapples with problems that are of paramount importance to political scientists, in both practical and theoretical terms.

These problems include questions of how to apportion legislative seats to states and parties; how to measure concepts such as representation, fairness, and inequality; how to construct rules that are immune to strategic behavior, such as gerrymandering; how to design algorithms that allocate scarce resources like housing; and how to design stable constitutions (to name a few). And while the term social choice theory typically conjures to mind the process of aggregating individual preferences into a collective preference, the mathematical generality of social choice–theoretic approaches to aggregation renders many of the field's results directly applicable to topics far outside the realm of individual and collective preference. Examples include devising measures of network centrality, community detection algorithms, and ranking algorithms for search engines.

To introduce the reader to the scope of topics that social choice theory addresses—a literature that has been developed over the half century since Arrow published his result—we have chosen breadth over depth. We begin with a brief discussion of Arrow's theorem (Section 2) and quickly move to a large body of work that directly extends and speaks to Arrow's original results (Section 3). Section 4 examines properties of voting and constitutional systems. This large and well-known body of work attacks the problem of institutional design axiomatically and provides a number of results—both positive and negative—that characterize what our systems can accomplish and why. Section 5 moves away from the aggregation of preferences to survey work that

[1] Of course, other scholars published seminal results at the same time, including Duncan Black's median voter theorem (see sidebar titled Single-Peakedness and Black's Theorem) and Kenneth May's axiomatic characterization of majority rule (see Theorem 6, below).

tackles institutional design questions directly related to fairness, such as how to design systems of apportionment and geographical districting. Section 6 moves from the question of institutional design to the measurement of normative concepts of fundamental relevance to politics; we focus on justice and inequality. Finally, Section 7 discusses the applicability of social choice–theoretic concepts to the aggregation of data, both "big" and small, into numerical measures. We have structured the sections so that—after a brief introduction to our notation—readers can jump to any topic of interest. And we have tried to make our exposition broadly accessible to a nontechnical audience, while retaining enough formalization to make our points clearly.

We hope readers will draw two main conclusions from this review. First, Arrow's theorem tells us that, for many of the types of problems and applications outlined above, there is no correct answer (or, perhaps more accurately, there may be many correct answers). Second, the axiomatic approach utilized by Arrow provides a foundation for constructing and comparing the various possible ways to solve these problems. Accordingly, our hope is that this article will open a broader dialogue between theorists working on these types of problems and political scientists actively employing these measures and concepts in their research. At the very least, we hope that our survey of "social choice since Arrow" will provide political scientists with some insight into the properties of the measures and concepts they regularly utilize, and inspiration for devising and utilizing new measures. Before continuing, we note that we are of course not the first to survey these literatures. In addition to the many excellent reviews we cite throughout, we highly recommend the reviews by Plott (1976), Sen (1986), Austen-Smith & Banks (1999, 2004), and List (2013).

2. ARROW'S IMPOSSIBILITY THEOREM

The starting point of Arrow's theorem is a finite set of $k \geq 3$ alternatives that must be ranked by a group containing $n \geq 2$ people. (See sidebar titled Preferences as Criteria for an alternative interpretation of this setting.) We denote the set of alternatives by X and the set of individuals by N. Each person $i \in N$ ranks the alternatives in X according to an individual preference ordering, and this ranking is denoted $\succeq_i$ for person $i \in N$. If person i likes alternative $x \in X$ as much as or more than alternative $y \in X$, we write $x \succeq_i y$. If he likes x strictly more than y we write $x \succ_i y$, and if he is indifferent we write $x \sim_i y$. Each individual i is presumed to be rational in the sense that his preference relation is both complete (i.e., $x \succeq_i y$ or $y \succeq_i x$ or both) and transitive (i.e., if $x \succeq_i y$ and $y \succeq_i z$, then $x \succeq_i z$). Thus, the ranking $\succeq_i$ is a weak ordering of the alternatives in X, and the set of all weak orderings of X is denoted by $\mathcal{R}$. Finally, we denote the list, or profile, of all individuals' preferences by $\rho = (\succeq_1, \succeq_2, \ldots, \succeq_n)$.

Arrow's theorem focuses on preference aggregation rules. These are simply functions that take the preference profile ρ as an input and produce a collective preference relation, $\succeq$, that compares the alternatives. An arbitrary preference aggregation rule is denoted by f, so that $f(\rho)$ describes

PREFERENCES AS CRITERIA

Much of the discussion about Arrow's theorem has focused on the interpretation of the axioms within, and the implications of the result for, preference aggregation through democratic institutions, such as elections and legislatures. However, as we see in Sections 5 and 7, one can interpret any individual's "preferences" more generally as any ordering of the alternatives. That is, there is nothing in the axioms or the theorem that requires that the criteria being aggregated actually represent preferences. Taking a more general view of the inputs makes clearer the connections between Arrow's work and applied problems such as apportionment, gerrymandering, and data aggregation. This is discussed in greater detail by Patty & Penn (2014, ch. 3).

the group's preferences over the alternatives when the individual preferences are as described by ρ. We denote the preference relation returned by f at ρ by $\succeq_{f(\rho)}$, so that $x \succeq_{f(\rho)} y$ implies that x is ranked at least as highly as y by the social preference relation returned when the profile of individual preferences is $\rho \in \mathcal{R}^n$. Finally, Arrow requires that f return a social preference relation for all possible preference orderings ρ—this requirement is referred to as universal domain.

2.1. The Axiomatic Method

Arrow lays out four simple axioms that he argues any reasonable aggregation rule should satisfy. He then proves that these axioms are incompatible with each other; that no rule can simultaneously satisfy all four. In so doing, his result implies that any aggregation rule—regardless of what is being aggregated or for what purpose—must violate at least one of these axioms. Put differently, every democratic institution, be it electoral, legislative, administrative, or judicial in character, violates at least one of these axioms. We now define each of these four axioms in turn.

- **Pareto.** Arrow's first axiom requires that the aggregation rule be minimally responsive to the preferences of the individuals, in that it respects a unanimity condition. More formally, an aggregation rule f satisfies Pareto if, whenever every individual i strictly prefers x to y, the aggregation rule f ranks x strictly higher than y.
- **Independence of irrelevant alternatives.** Arrow's second axiom requires that the aggregation method should not consider irrelevant alternatives when comparing any pair of alternatives. Specifically, the independence of irrelevant alternatives (IIA) axiom requires that the group members' preferences about some alternative c not affect how the aggregation rule f ranks two different alternatives, $a \neq c$ and $b \neq c$.
- **Transitivity.** Arrow's third condition focuses on the ability of a preference aggregation rule to generate an unambiguous winner (or a collection of unambiguous winners, if there is a tie). An aggregation rule that generates the social ranking $x \succ y$, $y \succ z$, and $z \succ x$ (referred to as a cycle) does not provide an unambiguously "best" alternative when comparing x, y, and z. Arrow's transitivity axiom, requiring that f produce a transitive social ordering, rules out this possibility and more: If f produces an ordering in which $x \succeq y$ and $y \succeq z$, then it must also be the case that $x \succeq z$.
- **No dictator.** Arrow's final axiom requires that the preference aggregation rule be responsive to the preferences of more than one person. An aggregation rule f is "dictatorial" if there is one particular voter whose individual strict preferences always determine the social preference ordering, irrespective of the preferences of the other voters. Formally, if a rule is dictatorial then there exists one voter i such that every time $x \succ_i y$ (i.e., the dictator i strictly prefers x to y), the aggregation rule f produces a strict ranking $x \succ y$. An aggregation rule f satisfies *no dictator* if it is not dictatorial.

2.2. Arrow's Impossibility Theorem

With the four axioms of Pareto, IIA, transitivity, and no dictator defined and described, we are now ready to state Arrow's theorem (1951, 1963).

Theorem 1 (Arrow 1951, 1963). With three or more alternatives, any aggregation rule f satisfying universal domain, Pareto, IIA, and transitivity must be dictatorial.

Arrow's theorem tells us that if a group wishes to design a preference aggregation rule that is Pareto efficient, transitive, and independent of irrelevant alternatives, and if we place no restrictions on the preferences that individuals may have, then the rule must grant all decision-making authority to a single individual. Put another way, any aggregation rule that is not dictatorial must violate transitivity, Pareto, or IIA. We now turn to four important literatures that directly emerged in response to the theorem.

3. ARROW'S DESCENDANTS

In the 25 years following the first publication of *Social Choice and Individual Values*, several important threads of inquiry quickly emerged that extended the theorem to account for weakenings of Arrow's axioms. These included removing the Pareto principle (Wilson 1972) and replacing transitivity with weaker collective rationality requirements (Gibbard 2014, Brown 1975). In this section, we focus on three of these lines of work: altering IIA to allow for cardinal (as opposed to ordinal) preferences, allowing for strategic individual behavior through preference misrepresentation, and requiring the aggregation rule to respond to a richer set of information (in the case we consider, individual rights). Originally motivated to test the limits of Arrow's conclusions, these literatures succeeded in pinpointing many of the general mechanisms driving Arrow's result, and highlighted the sweeping generality of the theorem. We briefly review these early extensions in order to set the stage for subsequent developments.

3.1. Cardinal Preferences

A common criticism of Arrow's theorem was that it allows the preference aggregation rule f to respond only to ordinal information about voters' preferences, as the preferences of the individuals are expressed as orderings of the alternatives. Sen (1970a, pp. 118–30) partially answered this objection by showing that Arrow's result can be extended to allow for cardinal utilities. That is, instead of assuming that each individual simply ranks these alternatives, Sen's approach allows every individual $i \in N$ to assign a number to each alternative, $u_i(x)$, with $u_i(x) > u_i(y)$ meaning that i likes x more than y. This approach allows the aggregation rule to respond to the cardinality, or strength, of individuals' preferences over the alternatives.

To capture this possibility, we write $u = (u_1, \ldots, u_n)$ to denote a profile of utility functions, one for each individual. Then, letting $\mathcal{U} \equiv \mathbf{R}^X$ denote the set of all utility functions on X and $\mathcal{U}^n$ denote possible profiles of n utility functions on X, we define an aggregation functional as a mapping $F : \mathcal{U}^n \to \mathcal{R}$ that assigns each possible profile of utility functions to an ordinal ranking of the alternatives. Extending Arrow's theorem to a cardinal preference setting requires that we modify Arrow's axioms as follows.

- **Pareto.** An aggregation functional F satisfies Pareto if $u_i(x) > u_i(y)$ for all $i \in N$ implies that $x \succ_{F(u)} y$.
- **Cardinal IIA.** An aggregation functional F satisfies cardinal IIA if, for any pair of utility profiles, $u, v \in \mathcal{U}^n$, and any pair of alternatives $x, y \in X$ such that $u_i(x) = v_i(x)$ and $u_i(y) = v_i(y)$, $x \succeq_{F(u)} y \Leftrightarrow x \succeq_{F(v)} y$. In words, if individual utilities for x and y are unchanged, then the social ranking of x and y should be unchanged.
- **Transitivity.** An aggregation functional F satisfies transitivity if, for every utility profile $u \in \mathcal{U}^n$, $F(u)$ is a transitive ordering of the alternatives.
- **No dictator.** An aggregation functional F is dictatorial if there exists $i \in N$ such that $u_i(x) > u_i(y)$ implies that $x \succ_{F(u)} y$. F satisfies *no dictator* if it is not dictatorial.

Moving from ordinal to cardinal preferences requires one to consider how to compare different utility functions that are ordinally equivalent (i.e., that rank the alternatives in the same way). d'Aspremont & Gevers (2002) provide an excellent review of the large literature considering different ways to do this. However, due to space constraints, we focus only on the approach taken by Sen (1970a).

Individuals' utilities are cardinally measurable and noncomparable when any pair of utility functions $u_i, v_i \in \mathcal{U}$ are considered to be the same as (or equivalent to) each other if there is a real number $a_i \in \mathbf{R}$ and a positive real number $b_i \in \mathbf{R}_{++}$ such that $v_i(x) = a_i + b_i \cdot u_i(x)$ for all $x \in X$. Such an assumption implies that the magnitude of one person's utility (i.e., the size of differences in their utility between any pair of alternatives) is not comparable to another's. Then, an aggregation functional F is said to respect noncomparability if for any pair of profiles of utility functions $u, v \in \mathcal{U}^n$ for which there exists $(a_1, \ldots, a_n) \in \mathbf{R}^n$ and $(b_1, \ldots, b_n) \in \mathbf{R}^n_{++}$ such that $v_i(x) = a_i + b_i u_i(x)$ for all $i \in N$ and $x \in X$, $F(u) = F(v)$. With these definitions in hand, Sen (1970a) proves the following extension of Arrow's theorem.

Theorem 2 (Sen 1970a). Any aggregation functional F satisfying universal domain, Pareto, cardinal IIA, and transitivity while respecting noncomparability must be dictatorial.

Sen's theorem illustrates that Arrow's conclusions are not the result of ignoring cardinal information about (say) the strengths of individuals' preferences. Rather, as we will see in Section 6, Sen's result indicates that the impossibility theorem is about *comparability* of individuals' preferences. As Blackorby & Bossert (2008, p. 423) succinctly describe the situation, "[T]he most promising route of escape from the negative conclusion of Arrow's theorem is to consider informational environments that allow for interpersonal comparisons of well-being." We build on this point in a more applied fashion elsewhere, discussing the appeal and vulnerabilities of this approach (Patty & Penn 2015a, pp. 62–67). The principal weaknesses of taking this route are the Gibbard-Satterthwaite and Muller-Satterthwaite theorems, which are discussed next.

3.2. Strategic Behavior

Arrow's theorem is concerned with aggregation rules, which return social preference orderings. Of course, many democratic systems focus solely on producing a final winner or chosen outcome. Such institutions are more parsimoniously represented by a choice correspondence that maps each preference profile into a nonempty subset of the set of alternatives. Thus, letting $\mathcal{X}$ denote the nonempty subsets of X, we denote a choice correspondence by $C : \mathcal{R}^n \to \mathcal{X}$. When a choice correspondence C selects exactly one alternative for each preference profile (of course, the alternative it selects can differ across various preference profiles), then C is referred to as a choice function. For any choice function C, the number of choices possible under C is defined as the number of alternatives in X for which there exists at least one preference profile, $\rho \in \mathcal{R}^n$, at which C returns the alternative. [Formally, this is the cardinality of the following set: $\{x \in X : \exists \rho_x \in \mathcal{R}^n \text{ such that } C(\rho_x) = x\}$.] Finally, a choice correspondence C satisfies universal domain if it returns a nonempty set of alternatives for every possible preference profile (i.e., for all $\rho \in \mathcal{R}^n$, $C(\rho) \neq \emptyset$).

Regardless of whether we are considering aggregation rules or choice correspondences, a practical challenge for any attempt to make choices on the basis of individuals' preferences is how to elicit these preferences from the individuals. That is, real-world rules, such as voting systems, must rely on reported preferences (e.g., ballots) rather than individuals' true latent preferences. This line of research asks how and when an individual might have an incentive to misreport their preferences.

To represent this formally, suppose that each individual $i \in N$ reports a preference ordering, $b_i \in \mathcal{R}$ (where b stands for "ballot"). This report does not need to be truthful—individual i's truthful ballot is $\succeq_i$, defined previously. After all individuals have submitted their ballots,

SOCIAL CHOICE, REVELATION, AND MECHANISM DESIGN

Mechanism design refers to the study of how and whether incentives can be structured so as to induce individuals to achieve some goal(s) with respect to the individuals' choices. The mechanism that structures the incentives is essentially a game between the players, paired with a solution concept for that game, such as Nash equilibrium (Nash, Jr. 1950). Generally, the goals depend on information known only to the individuals (e.g., their individual preferences over alternatives, or "types"), so the general purpose of mechanism design is to incentivize individuals to reveal this information. A key result in this field, due in various ways to Gibbard (1973), Dasgupta et al. (1979), and Myerson (1979), is the revelation principle, which essentially states that it is sufficient to consider games in which the players only report their type (though not necessarily truthfully). An incentive-compatible mechanism is a game in which it is in the players' interests to report their types truthfully. Accordingly, an incentive-compatible mechanism is equivalent to a choice function. Because of this, incentive-compatible mechanisms—and what they can achieve or, in the parlance of the literature, implement—are described and constrained by the results of social choice (most famously, the Gibbard-Satterthwaite theorem and, by implication, Arrow's theorem). For a very clear discussion of this fundamental relationship, see Austen-Smith & Banks (1999, pp. 187–94).

$b = (b_1, \ldots, b_n)$, the choice function C selects the outcome, $C(b) \in X$. With this in hand, given any preference profile $\rho \in \mathcal{R}^n$, an individual $i \in N$ has an incentive to manipulate C at ρ if there exists a ballot $b_i \neq \succeq_i$ such that

$$C((\succeq_1, \ldots, \succeq_{i-1}, b_i, \succeq_{i+1}, \ldots, \succeq_n)) \succ_i C(\rho).$$

A choice function C for which there exists an individual $i \in N$ and preference profile $\rho \in \mathcal{R}$ such that i has an incentive to manipulate C at ρ is referred to as manipulable. Any choice function that is not manipulable is referred to as strategy-proof.

A strategy-proof choice function is consistent with every individual truthfully reporting (or "revealing"—see sidebar titled Social Choice, Revelation, and Mechanism Design) their preferences. If the choice function is manipulable, then there may be reason to suspect that one or more individuals misreported their preferences.

3.2.1. The Gibbard-Satterthwaite theorem.

The key result in this literature was proved independently by Gibbard (1973) and Satterthwaite (1975). Gibbard and Satterthwaite demonstrate that if at least three different outcomes are possible—so that the range of the choice function contains at least three elements—dictatorial choice functions are the only ones that are strategy-proof. In other words, there is no nondictatorial choice function that is strategy-proof.

Theorem 3 (Gibbard 1973, Satterthwaite 1975). With universal domain and three or more outcomes possible, any strategy-proof choice function is dictatorial.

Interpreting choice functions as voting systems, the Gibbard-Satterthwaite theorem proves that strategic voting (i.e., the possibility of benefiting from voting against one's true preferences) is endemic to all deterministic voting systems when there are at least three alternatives that can be chosen. Of course, the possibility that individuals might have an incentive to cast an insincere ballot is well known—the surprising aspect of the Gibbard-Satterthwaite theorem is its scope. Specifically, every voting system can create an incentive to vote insincerely unless (*a*) there is only one voter whose vote matters or (*b*) there are only one or two alternatives that the system can select or (*c*) certain individual preferences are disallowed (i.e., universal domain is violated).

MONOTONICITY AND IMPLEMENTABILITY OF CHOICE CORRESPONDENCES

A fundamental result connecting social choice theory and mechanism design is due to Maskin (1999) (originally circulated in 1977). A choice correspondence C is implementable in Nash equilibrium if there exists a game $\gamma : \mathcal{U}^n \to X$ such that, for all $u \in \mathcal{U}^n$, $C(u) = NE_\gamma(u)$, where $NE_\gamma(u)$ denotes the set of Nash equilibrium outcomes for game γ, given the utility profile u. If C is implementable in Nash equilibrium, then there exists a game for which the Nash equilibrium outcomes of the game correspond to the outcomes prescribed by C. Maskin's theorem establishes that a very strong monotonicity condition is necessary for implementability in Nash equilibrium, and consequently results concerning Nash implementation are generally negative. Refining the equilibrium concept—for example, to undominated Nash—leads to more permissive results (Palfrey & Srivastava 1991).

3.2.2. The links between Arrow and Gibbard-Satterthwaite. Despite the differences between the settings for the two theorems, the Gibbard-Satterthwaite theorem and Arrow's theorem are mathematically similar. As an illustration of this, Reny (2001) presents two clear and side-by-side proofs, one for Arrow's theorem and one for the Gibbard-Satterthwaite theorem, showing that the two results can be proved using the same constructive method. Similarly, though somewhat more abstractly, Eliaz (2004) also demonstrates how the two theorems can be derived from a common and more general "metatheorem" on social aggregators.

3.2.3. The Muller-Satterthwaite theorem. Shortly after Gibbard (1973) and Satterthwaite (1975) proved their theorems, Muller & Satterthwaite (1977) proved that when individual preferences are strict (so that no individual is indifferent between any two alternatives), strategy-proofness of a choice function C is equivalent to an intuitive, but strong, monotonicity condition (see sidebar titled Monotonicity and Implementability of Choice Correspondences). Specifically, a choice function C satisfies *strong positive association* if, for any pair of ballot profiles $b, b' \in \mathcal{R}^n$ and any pair of alternatives $x, y \in X$ such that every voter who prefers x to y in ρ also prefers x to y in ρ' (i.e., $x \succ_{b_i} y \Rightarrow x \succ_{b'_i} y$), $C(\rho) = x$ implies $C(\rho') = x$. In other words, if x is chosen from one ballot profile b, and another profile b' exists in which x has not declined relative to any other alternative y for any individual, strong positive association requires that x also be chosen at b'. The equivalence of strategy-proofness and strong positive association yields the following corollary of the Gibbard-Satterthwaite theorem.

> **Theorem 4 (Muller & Satterthwaite 1977).** With (strict) universal domain and three or more outcomes possible, any choice function satisfying strong positive association is dictatorial.

3.3. Individual Rights

Arrow's axioms are a sparse representation of the rich process of collective decision making, and many of the first criticisms Arrow faced when presenting his result concerned his neglect of other, normatively essential, values. This, in combination with debates about the IIA axiom, prompted scholars to develop and incorporate alternative desiderata into the framework. One thread of this work is concerned with how to bring individual liberty and/or rights into the Arrovian framework. We now present a seminal contribution to this literature. Suzumura (2011) provides a thorough review of these early debates and the progression of social choice approaches to individual rights.

To capture the notion of an individual i's rights within a social choice framework, suppose that two alternatives, x and y, differ only with respect to features that are private to i, where "private to

i" describes aspects of the alternatives about which no individual other than i should be accorded a say. In this case, the choice between x and y is said to be within i's protected sphere, denoted by $D_i \subseteq X \times X$ [for obvious reasons, if $(x, y) \in D_i$, then $(y, x) \in D_i$ as well]. Thus, an individual's protected sphere is a collection of pairs of alternatives between which that person's preference should be respected (i.e., protected). Each individual i's set of protected spheres is exogenously given—they represent the context of the social choice problem. Given any individual i, a social choice correspondence C is said to respect i's individual rights if, for all preference profiles $\rho \in \mathcal{R}^n$,

$$\left[(x, y) \in D_i \text{ and } x \succ_i y\right] \Rightarrow y \notin C(\rho).$$

With this notion of protecting rights in hand, Sen (1970b) defines the following two axioms for choice correspondences.

- **Minimal liberalism.** A choice correspondence C respects minimal liberalism if there are at least two individuals, $i, j \in N$, for whom $D_i \neq \emptyset$ and $D_j \neq \emptyset$, such that C respects both i's and j's individual rights.
- **Pareto.** A choice correspondence C satisfies Pareto if, for all preference profiles $\rho \in \mathcal{R}^n$, if $x \succ_i y$ for all $i \in N$, then $y \notin C(\rho)$.

Theorem 5 (Sen 1970b). There exists no social choice correspondence C satisfying universal domain, minimal liberalism, and Pareto.

Sen's theorem has broad implications. First, it is an impossibility result requiring neither transitivity nor IIA. Many, if not most, critiques of Arrow's theorem have aimed at one or both of those axioms. Sen's result shows that we can replace both with a different normatively defensible axiom and still obtain a contradiction with Pareto.

This raises the second implication of Sen's theorem. Specifically, the result presents a challenge to the Pareto principle itself and, accordingly, all welfarist theories of justice and social choice (see Section 6): Any social choice correspondence C satisfying universal domain and Pareto can respect the individual rights of at most one individual. In other words, the theorem demonstrates that protecting the rights of multiple individuals is inherently at odds with even the most minimal notions of utilitarianism. Or, equivalently, it illustrates that liberalism and utilitarianism can be mutually consistent only if one restricts the set of permissible individual preferences.

Sen's inquiry was subsequently extended by Gibbard (1974), who proves a generalization of Sen's theorem within a richer framework of choices that explicitly captures the notion of social outcomes resulting from both private and public decisions. Space limitations preclude a full presentation of Gibbard's result, but a short description is warranted. Gibbard's result complements Sen's in two important ways. First, Gibbard's definition of rights protection is more concrete than is Sen's—it answers some objections to Sen's notion of minimal liberalism [see Suzumura (2011), sections 3 and 4, for discussion of these objections]. Second, Gibbard's result does not require the Pareto axiom and is therefore, on its face, not necessarily inconsistent with welfarism.

4. VOTING AND CONSTITUTIONAL DESIGN

Preference aggregation rules are conceptually similar to voting procedures in which ballots (preferences) are forced through an algorithm that chooses either a winner or a social ranking of candidates. Given that a vote is the most natural way of eliciting a collective preference, it is not surprising that many social choice–theoretic results concern voting systems. Using the terminology from above, we conceive of a voting system as a choice correspondence C that maps a preference profile into a set of outcomes (if the set is multivalued, the elements in it are considered "tied"). The

SINGLE-PEAKEDNESS AND BLACK'S THEOREM

Formal models of voting often assume that individual preferences are single-peaked. This assumption presumes that there is some underlying ordering of the alternatives, $x_1 < x_2 < \cdots < x_k$, such that each individual i has a favorite policy (ideal point) $x^i = x_t$, and that person i's preferences are strictly decreasing as policies move away from i's ideal point: $x^i \succ_i x_{t+1} \succ_i x_{t+2} \ldots$ and $x^i \succ_i x_{t-1} \succ_i x_{t-2} \ldots$. In this setting, Black's median voter theorem (Black 1948) implies that a majority vote over pairs of alternatives yields a transitive social preference ordering when there is an odd number of voters (and a quasitransitive ordering when n is even). Furthermore, Black proves that any median of the individuals' ideal points (ordered according to the underlying ordering of alternatives) is weakly majority preferred to all other alternatives.

Gibbard-Satterthwaite theorem tells us that every (single-valued) voting system is manipulable (for multivalued systems, see Duggan & Schwartz 2000), and the Muller-Satterthwaite theorem tells us that every single-valued voting system violates a natural monotonicity condition, but much of this literature is more constructive, characterizing voting systems by the axioms they do and don't satisfy. A more detailed review is provided by Brams & Fishburn (2002).

4.1. Majoritarian Methods of Voting

The most classic result on voting procedures is by May (1952), who axiomatized plurality rule over pairs of alternatives. Specifically, May considered the following three axioms.

- **Anonymity.** A choice correspondence C satisfies anonymity if it treats all individuals equally.
- **Neutrality.** A choice correspondence C satisfies neutrality if it treats all alternatives equally.
- **Positive responsiveness.** A choice correspondence C satisfies positive responsiveness if whenever an alternative x is winning or tied for the lead under one profile, ρ [i.e., $x \in C(\rho)$], and under a different profile, ρ', support for x has only increased among the voters, then x must be the unique winner under ρ' [$x = C(\rho')$].

Theorem 6 (May 1952). When $|X| = 2$, plurality rule uniquely satisfies anonymity, neutrality, and positive responsiveness.

While May (1952) and Black (1948) provided important defenses of majority and plurality rules (see sidebar titled Single-Peakedness and Black's Theorem), Dasgupta & Maskin (2008) more recently refined a long literature into a very specific defense of majority rule. They begin with a result similar to Arrow's theorem, proving that no voting rule satisfies Pareto, anonymity, IIA, neutrality, and decisiveness (the requirement that the rule be discriminating in the sense of picking a unique winner). However, like Arrow's theorem, this result depends on the assumption of universal domain. There are smaller classes of preference domains, such as the domain of single-peaked preferences, described next, in which aggregation and voting rules can be designed to satisfy all of these axioms. Dasgupta & Maskin (2008) then show that majority rule is the unique voting rule that works well (in the sense of satisfying the above axioms) on the biggest class of preference domains.

4.2. Positional Methods of Voting

Majority rule is generally considered desirable from an axiomatic perspective when preferences are single-peaked, but it presents practical limitations. Taking a vote over all pairs of alternatives is logistically challenging, as is requiring individuals to submit entire rankings of the alternatives in

VOTING PARADOXES

A voting paradox occurs when a seemingly sensible voting procedure produces undesirable outcomes. Much of the axiomatic literature on voting systems concerns a procedure's susceptibility to paradoxes, and violations of many of the axioms we have considered thus far can be reformulated as paradoxes. Some other well-studied examples are the following:

- **The no-show paradox**. By choosing to abstain from voting, a voter can obtain a strictly better outcome for himself.
- **The preference inversion paradox**. Reversing all individuals' preference orderings—making each person's last choice first, and so on—leaves the social ranking of alternatives (or unique winning alternative) unchanged.
- **Condorcet (in)consistency**. A Condorcet winner (an alternative that defeats every other alternative in a head-to-head contest) exists, but the rule fails to select it.
- **The additional support paradox**. All else equal, additional support for an alternative causes that alternative to lose.

Nurmi (1999) provides an excellent and extensive review of voting paradoxes and "how to deal with them."

question. A family of voting rules that potentially offer a compromise between these two extremes are the positional methods of voting. A positional voting method asks each voter to strictly rank all or some of the candidates under consideration. For each voter's ballot, the method then assigns a score to each candidate, depending on the voter's ranking. Given all of the voters' ballots, the final overall ranking of the candidates is determined by tallying the scores. These methods differ in the particular scores they assign to each position. Plurality rule, one of the most commonly used voting systems in the world, assigns a score of 1 to the top-ranked candidate on each voter's ballot and 0 to all other candidates. Borda count, another well-known system, assigns $k - 1$ points to each ballot's top-ranked candidate, $k - 2$ to each second-ranked candidate, and so on.

Positional methods are guaranteed to yield a transitive social preference ordering, because they assign point totals to each alternative. However, Saari (1989) shows that even within this class of rules there is ample opportunity for potentially unwelcome outcomes (see sidebar titled Voting Paradoxes). For example, all positional rules suffer from the no-show paradox (Saari 2011), and all positional rules are Condorcet inconsistent (Gärdenfors 1973). A number of authors have analyzed these rules axiomatically and have shown that some surprising properties are uniquely satisfied by the Borda count among the class of positional rules (see Gärdenfors 1973, Young 1974, Saari 1990, 2000, Saari & Barney 2003, Emerson 2013).

4.3. Multistage and "Multiple" Rules

Multistage voting procedures can utilize either ranked ballots (such as the alternative vote) or unranked, categorical ballots (such as plurality rule with runoff). In both cases, candidates with the lowest levels of support are sequentially eliminated. Typically, the voters who had initially supported a losing candidate are able to transfer their vote to a different candidate, either in a second round of voting or during the vote tallying process. Despite the popularity of these systems, it is well known that they are vulnerable to the additional support paradox, a particularly noxious type of monotonicity violation. The problem occurs because first-place votes affect the order of elimination; increased support for a winner can alter which challenger that winner faces in round 2, thus turning the winner into a loser. This problem was first identified by Doron & Kronick (1977).

Brams & Fishburn (1984) demonstrate different problems that can arise if some voters truncate their rankings; under transferable vote systems, voters may materially benefit by submitting a truncated ballot.

Under a multiple rule (Saari & Van Newenhizen 1988), a voter "ranks the candidates and then selects a positional rule to tally the ballot from two or more choices" (Saari 2010, p. 218). The most well-known examples of such rules are the approval vote and the cumulative vote. Under approval voting, a voter approves or disapproves of each candidate, and candidates receive one point for each approval. Under the cumulative vote, voters may distribute an integer number of votes across the candidates. Casella (2005) has introduced a voting system similar to the cumulative vote, but over pairs of alternatives in multiple elections. She demonstrates that strategic behavior induced under her system of storable votes can generate minority victories, and that the system can help elicit voter preference intensities. A similar system termed qualitative voting has been studied by Hortala-Vallve (2012).

Nonstrategic analysis of multiple rules can pose challenges, because different voters with the same preference orderings may choose to mark their ballots differently (Saari 2011). Brams and Fishburn have written much on the approval vote and have independently and jointly proved that it satisfies a number of desirable properties. Many of these results assume a restricted domain of preferences termed dichotomous preferences, in which each voter partitions the set of alternatives into "good/acceptable" and "bad/unacceptable" categories. Brams & Fishburn (1978) show that approval voting equals the Condorcet rule (a rule choosing the Condorcet winner or winners) when preferences are dichotomous. Building on work by Fishburn (1978), Vorsatz (2007) shows, among other things, that a choice correspondence satisfies anonymity, neutrality, strategy-proofness, and monotonicity if and only if it is the approval vote. He interprets his result as an extension of May's theorem.

4.4. Constitutional Consistency and Self-Selectivity

So far we have discussed voting procedures in terms of their ability to satisfy certain axioms regarding their responsiveness to individual preferences (e.g., anonymity, neutrality, monotonicity, Condorcet consistency). A different line of work steps back to analyze the choice of procedure itself and asks whether a procedure that is chosen to choose among procedures that will choose among alternatives will choose itself. This self-referential consistency property was notably characterized by Koray (2000), who showed that with sufficiently rich preferences, the only self-selective choice functions are dictatorial (thus providing a different route to the Arrow and Gibbard-Satterthwaite theorems). Other studies have explored similar themes, relaxing or altering Koray's framework to bypass his impossibility result. Barberà & Jackson (2004) use a weaker notion of self-stability in a different choice environment, whereby under a self-stable constitution, society would not vote to change the constitution. They show that such rules may or may not exist and that majority rule possesses some special properties with respect to self-stability. Koray & Slinko (2008) extend Koray's original result, finding that under a suitably restricted environment, nondictatorial self-selective rules exist. Alternative approaches can be found in the work of Barberà & Beviá (2002), Houy (2004), and Bhattacharya (2018).

5. FAIRNESS: APPORTIONMENT AND GERRYMANDERING

There is a large literature concerned with axiomatic measures of fairness (Thomson 2011 provides an excellent review). We focus here on two important practical problems regarding political fairness: apportionment and gerrymandering.

5.1. Apportionment

A fundamental and practical question in democratic institutional design is how to apportion seats in a legislative body. This question comes up in multiple forms—for example, in proportional representation systems, seats must be apportioned across the various political parties based on the number of votes they receive, and in district-based systems, seats must be apportioned across the districts.

The basic problem can be represented as follows. Each of $s \geq 3$ states (or parties) has a population (or vote) share denoted by $p_i \in [0, 1]$ for $i \in \{1, \ldots, s\}$ with $\sum_{i=1}^{s} p_i = 1$. We denote the vector of all population shares by $p = (p_1, \ldots, p_s)$. Based on p, each of the s states must be assigned an integer number of seats, denoted by $a_i \in \{0, \ldots, H\}$, where $H \geq 1$ is an integer representing the total number of seats to be apportioned. Letting P^s denote the set of all population shares for the s states (i.e., the $s - 1$ dimensional simplex), we define an apportionment method as a correspondence $A : P^s \times \mathbf{Z}_{++} \rightrightarrows \{0, 1, \ldots, \infty\}^s$ such that all seats are assigned: $\sum_{i=1}^{s} M(p, H)_i = H$.

Multiple approaches to this problem have been proposed through the years. The seminal axiomatic contribution to apportionment was provided by Balinski & Young (2001), who defined two desirable properties for apportionment methods. The first of these, referred to as quota, requires that the method give each state its fair share of seats rounded either up or down, where "fair" is defined as the (typically noninteger) number of seats that would make the state's proportion of the legislature equal to its proportion of the population. Formally, quota is defined as follows.

Definition 1. An apportionment method M satisfies quota if, for all $p \in P^s$ and $H \in \mathbf{Z}_{++}$,

$$M(p, H)_i \in \left\{ \lfloor p_i H \rfloor, \lceil p_i H \rceil \right\}.$$

A second desirable property for apportionment methods is that if state i's population share goes up and state j's goes down, then state i does not receive fewer seats while state j receives more seats. This property, known as population monotonicity, is formally defined below.

Definition 2. An apportionment method M satisfies population monotonicity if, for all $p, p' \in P^s$, $H \in \mathbf{Z}_{++}$, and all pairs of states i, j,

$$\frac{p'_i}{p'_j} > \frac{p_i}{p_j} \Rightarrow \neg[a'_i < a_i \text{ and } a'_j > a_j].$$

Balinski & Young (2001) prove many results about apportionment, including the following seminal result.

Theorem 7 (Balinski & Young 2001). An apportionment method M satisfies quota only if it violates population monotonicity.

Balinski & Young's (2001) theorem is more general than simply categorizing methods for assigning seats in a legislature. It is a fundamental result regarding ineradicable conflicts between different notions of fairness. The authors delve into a variety of other paradoxes that apportionment methods are vulnerable to, including the new states paradox (adding a new state and corresponding seats for that state can alter the seat distribution of other states), and the well-known Alabama paradox (if the total number of seats in a legislature is increased, no state's number of seats decreases). The new state paradox is similar in spirit to violations of IIA, while the Balinski-Young theorem establishes a tension between monotonicity and fairness requirements, in line with the Muller-Satterthwaite theorem.

5.2. Gerrymandering and Districting

Gerrymandering, or the drawing of geographical district lines to shape legislative representation, has long attracted attention in indirect democracies. It is a specific example of a more general possibility in which citizens or criteria are divided into subgroups prior to the application of an intermediate aggregation process (or processes) to produce a set of what one might refer to as representatives from each of the subgroups that are then used to produce a final choice or ranking of the alternatives. [Tasnádi (2011) provides a recent review of the literature on the question of districting.]

Chambers (2008, 2009) uses an axiomatic approach to consider and construct "gerrymander-proof" choice rules, while Bervoets & Merlin (2012) show that any gerrymander-proof system that is responsive to voters' preferences must essentially give every voter a veto. In both cases, the gerrymander-proof systems violate neutrality, meaning that the rule must privilege some alternatives over others.

Moving beyond gerrymandering, Puppe & Tasnádi (2015) consider general districting procedures. Adopting a normative stance, they present five axioms for districting procedures, each of which represents an aspect of what one might refer to as a fair districting procedure in a two-party system. They prove that any districting procedure that is anonymous in the sense of treating both parties the same must either use information beyond the number of seats won by the parties or violate a consistency condition requiring that, if two or more districts are combined and the districting procedure is applied to their union, the procedure would divide the union into the same districts as it originally returned.

6. MEASURING LATENT CONCEPTS: JUSTICE AND INEQUALITY

Measurement in the social sciences typically refers to the translation of a qualitative concept (such as well-being, proportionality, power, or representation) into a numerical quantity. Good measurement involves precisely defining mathematical objects and relations among them in order to meaningfully reflect empirical quantities and relationships of interest (Roberts 1984, p. xviii). Following Arrow's attempts to measure social welfare, the modern mathematical theory of measurement focuses on axioms or conditions that are necessary or sufficient for measurement to be possible. In this section, we consider axiomatic approaches to the measurement of justice and inequality.

6.1. Justice

The challenges in defining a social preference ordering apply equally to defining a measure of justice. Axiomatic theories of justice are divided into two families, welfarist and nonwelfarist. We focus here solely on welfarist theories, which depend only on individuals' utilities (i.e., welfares) and have attracted much more attention from social choice theorists.

Generally speaking, welfarist theories of justice relax IIA by explicitly incorporating cardinal notions of individual utility and relaxing, to various degrees, the assumption of noncomparability of individuals' utilities. This literature begins with the work of Sen (1970a) on extending Arrow's theorem to cardinal utilities, discussed above. Space limitations preclude a full treatment of this broad and deep literature (see the review by Blackorby et al. 2002); instead, we present one representative welfarism theorem.

A social-evaluation function, $W : \mathcal{U}^n \rightarrow \mathcal{R}$, maps profiles of utility functions into rankings of the alternatives. Modern axiomatic works on justice (and social welfare) extend Sen's original work on cardinal preferences in multiple ways, but the principal distinction is that this literature considers various definitions of the degree to which one can directly compare two or more individuals' utilities for the various alternatives. In this setting, Arrow's IIA and Pareto axioms are represented as follows.

- **Binary independence of irrelevant alternatives.** A social-evaluation function W satisfies binary independence of irrelevant alternatives if, for all $u, v \in \mathcal{U}^n$ and all $x, y \in X$, $u(x) = v(y)$ and $x \succeq_{W(u)} y$ implies $x \succeq_{W(v)} y$.
- **Pareto indifference.** A social-evaluation function W satisfies Pareto indifference if, for all $u \in \mathcal{U}^n$ and $x \in X$, $u(x) = u(y)$ implies that $x \sim_{W(u)} y$.

To formally represent how one might compare utility functions, we partition the set of utility profiles, $\mathcal{U}^n$, using a correspondence, $\Phi : \mathcal{U}^n \rightrightarrows \mathcal{U}$, satisfying the following equivalence requirement:

$$v \in \Phi(u) \Rightarrow u \in \Phi(v). \qquad 1.$$

The basic idea of this correspondence Φ is that any two utility profiles, $u, v \in \mathcal{U}$, with $u \in \Phi(v)$ and $v \in \Phi(u)$, must be deemed equivalent to each other in the sense that the function must return the same ordering of the alternatives. In general, a notion of comparability that divides $\mathcal{U}$ into fewer categories (i.e., Φ generates a coarser partition of $\mathcal{U}$) is weaker than one that divides $\mathcal{U}$ into more categories (i.e., a finer partition of $\mathcal{U}$). Regardless of which notion of comparability one adopts [e.g., see the reviews by d'Aspremont (1985) and Blackorby & Bossert (2008)], its impact is captured by the following informational requirement imposed on the social-evaluation function, W:

Definition 3. A social-evaluation function W satisfies information invariance with respect to Φ if, for all pairs of profiles of utility functions, $u, v \in \mathcal{U}^n$ N with $u \in \Phi(v)$, it is the case that $W(u) = W(v)$.

The following "welfarism theorem" demonstrates that any welfarist theory of justice must satisfy both independence and Pareto, and any theory of justice that satisfies both independence and Pareto is welfarist.

Theorem 8 (d'Aspremont & Gevers 1977, Hammond 1979). For any correspondence $\Phi : \mathcal{U}^n \rightrightarrows \mathcal{U}$ satisfying the partition requirement in Equation 1, a social-evaluation function W satisfying information invariance with respect to Φ and universal domain satisfies binary independence of irrelevant alternatives and Pareto indifference if and only if, for all $u \in \mathcal{U}$ and $x, y \in X$, the ordering of x and y under W depends only on $u(x)$ and $u(y)$.

Simply put, Theorem 8 implies that welfarism—the principle that justice should be based on individuals' welfares—is equivalent to IIA and Pareto efficiency. This is powerful for a couple of reasons, but perhaps the most surprising is that adopting a welfarist approach compels one to accept IIA (Patty & Penn 2019). That said, a subtle point regarding Theorem 8's interpretation is the role of the conjunction of binary independence and Pareto indifference. It is well known that there are social-evaluation functions that satisfy either one of the two that do not depend solely on information about individuals' welfares (see Blackorby & Bossert 2008).

6.2. Inequality

The systematic study of how to measure inequality dates back at least to Lorenz (1905), Gini (1912, 1921), Pigou (1912), and Dalton (1920), but its modern, axiomatic study arguably begins with Atkinson (1970) and Sen (1973). There are two links between this research program and the Arrovian project. First, higher inequality is (often) equated with lower social welfare, so that this

MEASURING INEQUALITY: LORENZ CURVES AND THE GINI COEFFICIENT

For any distribution of income, x, with cumulative distribution $F_x : \mathbf{R}_+ \to [0,1]$ and mean $\bar{x} > 0$, the Lorenz curve for x is defined as the following mapping $L_x : [0,1] \to [0,1]$:

$$L_x(p) = \frac{1}{\bar{x}} \int_0^p F_x^{-1}(t)\, \mathrm{d}t, \text{ for all } p \in [0,1].$$

For two distributions, x and y, we say that x has equal or higher inequality than does y under the Lorenz criterion, denoted by xLy, if

$$xLy \Rightarrow L_x(p) \leq L_y(p) \text{ for all } p \in [0,1].$$

An inequality measure I is Lorenz consistent if

$$xLy \Rightarrow xIy.$$

Finally, the Gini coefficient for x is

$$G(x) = \frac{1}{2\bar{x}} \iint |y - x|\, \mathrm{d}F(x)\, \mathrm{d}F(y).$$

Note that $G(x) > G(y)$ if xLy, but the converse need not hold: The Gini coefficient respects—but additionally completes (i.e., imposes additional structure above and beyond)—the Lorenz partial ordering.

project may be interpreted as searching for a well-behaved social welfare function in a restricted domain. (There are several ways to establish this link. Arguably, the most prominent approach is utilitarian and presumes that individuals share a common, strictly concave utility function over incomes.) Second, the fact that there exist multiple reasonable measures of inequality led to considerable work on the common and distinctive axiomatic foundations for various such measures.

A common (partial) ranking of two income distributions in terms of inequality is based on the Lorenz criterion, which serves as the basis for the widely used Gini coefficient (see sidebar titled Measuring Inequality: Lorenz Curves and the Gini Coefficient). There are four foundational axioms of relative inequality measures. The first is known as the Pigou-Dalton transfer principle, which requires the inequality measure to judge the inequality of a society as higher if income is transferred from a poorer person to a richer person. The second is symmetry, which requires that the inequality of a society be unchanged if the incomes are reassigned to different individuals but otherwise left unchanged. The third is mean independence, which requires that the measure be invariant to the units in which incomes are measured. The fourth is replication invariance, which is a very minimal requirement ensuring that the measure is not sensitive to changes in the size of the population that leave the distribution of incomes otherwise unchanged.

- **(PD) Pigou-Dalton transfer principle.** *Any transfer of income from a poorer person to a richer person must increase the inequality measure.*
- **(S) Symmetry.** *If individuals' incomes are simply reassigned within the society, then society's inequality remains unchanged.*
- **(M) Mean independence.** *If every individual's income is multiplied by a common factor* $\alpha > 0$, *then society's inequality remains unchanged.*
- **(R) Replication invariance.** *If society is replicated* $m \geq 1$ *times, then the inequality of the resulting society is the same as that of the original society.*

A key result in the measurement of inequality is that the inequality measures that satisfy PD, S, M, and R are exactly the Lorenz-consistent inequality measures:

Proposition 1 (Foster 1985). A relative inequality measure, I, satisfies PD, S, M, and R if and only if I is Lorenz consistent.

The Gini coefficient is Lorenz consistent and, accordingly, satisfies PD, S, M, and R. It is not the only inequality measure that satisfies these axioms, of course, because the Lorenz ranking is only a partial order; many income distributions are not comparable according to the Lorenz criterion. Thus, Foster's (1985) result clarifies where an inequality measure is "making choices" independent of PD, S, M, and R. Specifically, whenever an inequality measure ranks two income distributions that are not comparable according to the Lorenz criterion, the measure is imposing some additional criteria above and beyond the four axioms.

7. BIG DATA: CLUSTERING, MATCHING, AND NETWORKS

As data sets have become larger and more complex, and the cost of computational power has decreased, interest has grown in methods for summarizing and analyzing high-dimensional phenomena, such as networks, F-MRI (functional magnetic resonance imaging) scans, images, texts, and genetic information.

As we have written elsewhere, determining whether a particular data set constitutes "big" data is "a function of the data's underlying conceptual structure. Much more than being a function of the number of observations or the number of variables, data is 'big' if the concepts underlying the data—the data's *raison(s) d'être*—are more complicated than a list of vectors of numbers" (Patty & Penn 2015b, p. 95). In this section, we describe axiomatic approaches to three popular big data applications that reflect this attention to conceptual structure rather than simply size: clustering algorithms, matching procedures, and network measures. We then conclude with a very brief overview of emerging topics for axiomatic studies related to organizing and analyzing big data.

7.1. Clustering

The structure of big data is often not known a priori. Often, theory suggests that the data can and should be broken into distinct groups, where members of the same group are more similar to each other than they are to members of other groups. A popular class of methods to uncover such groupings are known as clustering algorithms, which attempt to infer groups (i.e., clusters) from a given data set. Many such algorithms exist, but the justifications for choosing one over another have often either been ignored or based on ad hoc considerations determined by the data in question. Kleinberg (2003) presents a set of axioms for clustering algorithms. A clustering problem is described by a set of $n \geq 2$ observations, $S = \{1, 2, \ldots, n\}$, and a distance matrix, $d : S \times S \to \mathbf{R}_+$ satisfying (1) $i = j \Leftrightarrow d(i, j) = 0$ and (2) $d(i, j) = d(j, i)$. Thus, one can think of d as a $n \times n$ nonnegative, symmetric matrix with zeros on the diagonal (and only on the diagonal).

A clustering function is a function, κ, that maps each distance matrix into a partition of S. [That is, for any distance matrix d, $\kappa(d)$ is a set of nonempty subsets of S, $\kappa(d) \subset 2^S$, such that, for all $\gamma, \gamma' \in f(d)$ $\gamma \cap \gamma' = \emptyset$ and $\cup\{\gamma \in \kappa(d)\} = S$.] For any partition of S, Γ and any pair of distance matrices d and d', d' is a Γ-transformation of d if it satisfies the following:

- if $i, j \in S$ are in the same set within Γ, then $d'(i, j) \leq d(i, j)$,
- if $i, j \in S$ are not in the same set within Γ, then $d'(i, j) \geq d(i, j)$.

With this notion of transformations in hand, Kleinberg (2003) defines the following three axioms for cluster functions:

1. **Scale invariance.** For any distance matrix d and any $\alpha > 0$, we have $\kappa(d) = \kappa(\alpha \cdot d)$.
2. **Richness.** For every partition of S, Γ, there exists some distance matrix d_Γ such that $\kappa(d_\Gamma) = \Gamma$.
3. **Consistency.** For any pair of distance matrices, d and d', if d' is a $\kappa(d)$-transformation of d, then $\kappa(d') = \kappa(d)$.

Scale invariance requires that the scale of units should not matter—arguably, it is uncontroversial, but it is violated by some threshold-based clustering algorithms. Richness is a regularity condition similar in spirt to both universal domain and Pareto in Arrow's original framework. Among other things, richness rules out a clustering function that always puts all observations into one cluster. Consistency is more substantively oriented. In casual terms, it requires that, if every cluster becomes more tightly clustered and all clusters become more distant from each other, then the resulting clusters returned by the clustering function should not change. Kleinberg (2003) proves that there is no clustering function that satisfies all three of these axioms.

Theorem 9 (Kleinberg 2003). There is no clustering function κ that satisfies scale invariance, richness, and consistency.

Linking Kleinberg's theorem with Arrow's IIA axiom and noncomparability, adopting richness implies that scale invariance is analogous to noncomparability (as defined and discussed in Section 3), and that consistency is analogous to IIA. Thus, one important practical implication of Kleinberg's theorem mirrors that of the d'Aspremont-Gevers-Hammond theorem (Theorem 8, above): Without some a priori restriction on the range of possible clusters, there is a fundamental and inescapable tension between two closely related measurement (or invariance) axioms. In a nutshell, satisfying richness implies that a clustering function must be sensitive to the units of the distance metric.

Extending this literature, Meila (2005) explores various axioms for clustering functions, and Ackerman & Ben-David (2009) consider axiomatizing measures of quality for clustering functions. Finally, Zadeh & Ben-David (2009) propose a slight modification to Kleinberg's (2003) richness axiom, k-richness, which, for any fixed $k \in \{1, \ldots, n\}$, requires the clustering function to return every possible set of k clusters. Replacing richness with k-richness, and imposing an additional axiom known as order consistency, Zadeh & Ben-David (2009) show that there is a unique clustering function, known as single-linkage clustering, that satisfies the four axioms.

7.2. Matching

Matching describes problems of assigning, say, individuals to discrete objects such as schools (Abdulkadiroğlu & Sönmez 2003), experimental treatments (Abdulkadiroğlu et al. 2017), legislative committees (Shepsle 1975), colleges, or partners (Gale & Shapley 1962). The design of matching procedures is inherently axiomatic: Planners have some properties (i.e., axioms) they want their procedure to satisfy (such as efficiency, fairness, and/or strategy-proofness); after all, planners with no goals could just assign the individuals randomly. For reasons of space, we discuss the baseline case in which there are n individuals, $N = \{1, \ldots, n\}$, and n objects, $O = \{1, \ldots, n\}$. Each agent has a strict preference ordering $\succ_i$ over the objects.

One famous matching procedure is known as the top-trading cycle (TTC) mechanism [due to David Gale, and presented first by Shapley & Scarf (1974)], which works as follows. Assign each agent $i \in N$ a unique initial endowment, $e_i \in O$. After the allocation, each agent $i \in N$ indicates ("points at") the agent who possesses i's most-preferred object (this may be i, if i already has his or her most preferred object). Denote the "pointing at" relation by $\rightsquigarrow$: If agent i points at agent j, then we write $i \rightsquigarrow j$. There will exist at least one cycle in the $\rightsquigarrow$ relation. Let $\{i_1, i_2, \ldots, i_m\}$ be such a cycle (i.e., $i_1 \rightsquigarrow i_2 \rightsquigarrow \ldots \rightsquigarrow i_m \rightsquigarrow i_1$). Assign each individual i in the cycle the object possessed by the agent i is pointing at. This is their final allocation, so remove them from the process. (Note that $i \rightsquigarrow i$ is a cycle, so that any agent who initially has his or her own most-preferred object is immediately removed in the first period.) With the remaining agents, repeat the process, so that now agents point at the remaining agent who possesses their most-preferred of the remaining objects. Continue until all agents are removed. Given that n is finite, this process will conclude after no more than n iterations.

A matching mechanism, in this context, is a function, $\mu : \mathcal{R}^n \to \mathcal{R}$, that maps preference profiles over O into an ordering over O. The produced ordering represents an assignment of the objects: If $N = \{1, 2, 3\}$ and $\mu(\rho) = 2 \succ 3 \succ 1$, this represents the mechanism assigning object 2 to individual 1, object 3 to individual 2, and object 1 to individual 3. In general, we write $\mu_i(\rho)$ to denote the object assigned to person $i \in N$ given the preference profile $\rho \in \mathcal{R}^n$.

In this setting, Ma (1994) characterized the TTC mechanism as the unique mechanism satisfying three axioms:

- **Individual rationality.** A matching mechanism μ satisfies individual rationality if for all $\rho \in \mathcal{R}^n$ and all $i \in N$, $\mu_i(\rho) \succeq_i e_i$.
- **Pareto.** A matching mechanism μ satisfies Pareto if for all $\rho \in \mathcal{R}^n$ and all distinct pairs of individuals $i \neq j \in N$, $\mu_j(\rho) \succeq_j \mu_i(\rho)$ or $\mu_i(\rho) \succeq_i \mu_j(\rho)$ (or both).
- **Strategy-proofness.** A matching mechanism μ satisfies strategy-proofness if for all $\rho \in \mathcal{R}^n$, all $i \in N$, and all $b_i \in \mathcal{R}$, $\mu_i(\rho) \succeq_i \mu_i((\rho_{-i}, b_i))$.

Individual rationality requires that no individual is ever made worse off than they were with their initial endowment, Pareto requires that the allocation be Pareto efficient, and strategy-proofness requires that it be a weakly dominant strategy for each individual i to truthfully reveal their preferences (i.e., to point at the agent who actually possesses the available object that i most prefers). Ma proves the following theorem.

Theorem 10 (Ma 1994). The TTC mechanism is the unique mechanism satisfying individual rationality, Pareto, and strategy-proofness.

Given that individual rationality and strategy-proofness are required for participation and truthfulness within the mechanism, and Pareto is necessary for there to be no gains from collusion between the individuals, Ma's (1994) result is particularly powerful in practical terms. Specifically, the result implies that the TTC mechanism is the only incentive-compatible mechanism for allocating indivisible objects across individuals without side-payments.

7.3. Networks

The study of networks has grown significantly over the past 20 years, particularly as the internet, search engines, and social media have become increasingly central to modern life. The use of networks, viewed as graphs, to capture notions such as power or influence in social situations has attracted sustained attention across the social sciences over the past 50 years (e.g., Myerson 1977).

7.3.1. Networks as graphs. Networks are often modeled as graphs. In this representation, the network consists of a set of n nodes, denoted by $N = \{1, \ldots, n\}$, and a set of edges (connections between nodes), denoted by $E \subset N \times N$. We denote a network by $G = (N, E)$. The set of all (directed) graphs on n nodes is denoted by $\mathcal{G}^n$. For any graph $G = (N, E) \in \mathcal{G}^n$ and distinct nodes $i, j \in N$, $G + (i, j) \equiv (N, E \cup \{(i, j)\})$ denotes the graph that results after the edge (i, j) is added to G and, for any permutation $\varphi : N \to N$, $\varphi(G) = (N, E')$ denotes the graph defined by $(i, j) \in E \Leftrightarrow (\varphi(i), \varphi(j)) \in E'$.

A centrality index is a function, $\sigma : \mathcal{G}^n \to \mathbf{R}^n$, that assigns to every node in a network a score that represents how central the node is relative to all other nodes in the network; the score assigned to node i in graph G is denoted by $\sigma(G)_i$. While a centrality index assigns a real number to each node (i.e, it is a cardinal measure), we can also think about the index as returning a ranking of the nodes (i.e., as an ordinal construct). For any centrality measure σ, any graph $G = (N, E)$, and any pair of nodes $i, j \in N$, the ordinal ranking of i and j according to σ is defined simply as $i \succeq_{\sigma(G)} j \Leftrightarrow \sigma(G)_i \geq \sigma(G)_j$ [and $i \succ_{\sigma(G)} j \Leftrightarrow \sigma(G)_i > \sigma(G)_j$].

Dozens of such indices have been defined and employed in the study of networks. Limited space precludes a lengthy treatment of such indices; the interested reader is referred to Borgatti & Everett (2006), Newman (2010), and Ward et al. (2011). A few high-profile indices are the following:

1. Out-degree centrality returns: for each node, the number of other nodes to which the node in question is connected. Formally, this is defined as $\sigma_i^d(G) \equiv |\{e \in E : e = (i, j)\}|$ for some $j \neq i$.
2. Betweenness centrality returns: for each node, the number of shortest paths between other pairs of nodes that contain the node in question.
3. Closeness centrality returns: for each node, the average length of the shortest paths from the node in question to every other node.

Recently, many scholars have begun developing and exploring potential axiomatizations of these and other centrality indices (e.g., Holzman 1990, Vohra 1996, Altman & Tennenholtz 2005, Boldi & Vigna 2014, Bandyopadhyay et al. 2017, and Boldi et al. 2017, to name only a few). For reasons of space, we focus on an elegant and powerful result presented by van den Brink & Gilles (2003), who define three simple axioms for centrality indices.

- **Anonymity.** A centrality index σ satisfies anonymity if for any G and any permutation $\varphi : N \to N$, $\sigma_{\varphi(i)}(\varphi(G)) = \sigma_i(G)$.
- **Positive responsiveness.** A centrality index σ satisfies positive responsiveness if for any $G = (N, E) \in \mathcal{G}^n$, any distinct triplet $i, j, k \in N$ with $(i, k) \notin E$, $i \succeq_{\sigma(G)} j$ implies that $i \succ_{\sigma(G+(i,k))} j$.
- **Independence of nondominated arcs.** A centrality index σ satisfies independence of nondominated arcs if, for all $G = (N, E), G' = (N, E') \in \mathcal{G}^n$ and $i, j \in N$ such that $(i, k) \in E \Leftrightarrow (i, k) \in E'$ and $(j, k) \in E \Leftrightarrow (j, k) \in E'$, $i \succeq_{\sigma(G)} j \Leftrightarrow i \succeq_{\sigma(G')} j$.

Anonymity is a standard requirement that implies the index can use only information about the edges of the graph when scoring the nodes. Positive responsiveness is another example of a monotonicity axiom and, as the name suggests, requires the index to be minimally responsive to adding edges to a graph. The independence of nondominated arcs axiom is less intuitive than positive responsiveness but is similar in the sense that, while positive responsiveness specifies what an index *should* measure, the independence axiom specifies what the index *should not* measure. In terms of desirability, the axiom—like most independence axioms—can be defended as ensuring that the index is robust to missing data (for more on this and other defenses of independence axioms, see Patty & Penn 2019).

The following theorem, due to van den Brink & Gilles (2003), shows that these three axioms exactly characterize centrality indices that are ordinally equivalent to out-degree centrality.

Theorem 11 (van den Brink & Gilles 2003). A centrality index σ satisfies anonymity, positive responsiveness and independence of nondominated arcs if and only if it is equivalent to out-degree centrality, σ^d. That is, for any graph $G = (N, E) \in \mathcal{G}^n$, and pair of nodes $i, j \in N$, $\sigma_i(G) \geq \sigma_j(G) \Leftrightarrow \sigma_i^d(G) \geq \sigma_j^d(G)$.

7.3.2. Other axiomatic approaches to the study of networks. Limited space precludes a deeper dive into axiomatic work regarding measurement of and within networks, but the axiomatic method is being applied broadly. For example, Butts (2000) develops axioms for the measurement of network complexity, Jin et al. (2011) provide an axiomatization of measures of similarity between roles in graphs, and Cohen & Zohar (2015) provide axiomatizations of link-prediction functions.

8. OTHER AREAS OF RESEARCH

As the breadth of this article presumably demonstrates, "social choice since Arrow" has generated, and continues to generate, axiomatic research into a wide-reaching array of topics. Indeed, the breadth is so great that we regrettably had to neglect some incredibly interesting and vibrant areas of research.

8.1. Data Aggregation and Compound Majority Paradoxes

Nurmi (1999, ch. 7) provides an excellent overview of compound majority and other voting paradoxes and how to deal with them (see also Lagerspetz 1996, Nurmi 1997, Nurmi & Meskanen 2000, Penn 2011). Compound majority paradoxes—such as the Anscombe, Ostrogorski, Referendum, and Simpson's paradoxes—each represent different challenges encountered when aggregation occurs multiple times and at different stages within an overall aggregation process. Examples of such processes are easy to find in politics. For example, votes are aggregated at the district (or party) level and into seats within a legislature, and then the resulting seats are aggregated into policy through the passage of legislation. Of course, these paradoxes have analogues in data analysis and inference—for example, comparing and combining the results of differently sized experimental trials.

8.2. Tournament Solution Concepts

A complete, but not necessarily transitive, binary relation on a set is referred to as a tournament. When using a nondictatorial preference aggregation rule that satisfies universal domain, Pareto, and IIA (e.g., majority rule), the collective choice problem must be defined over this set of (potentially cyclic) collective preferences. A large literature has developed over the past 50 years attempting to deal with such situations. Schwartz (1986) and Laslier (1997) are the standard references; more recent work in this area includes Moser et al. (2009), Penn (2006a,b), Fey (2008), and Duggan (2013).

8.3. Domain Restrictions

A recurring objection to Arrow's theorem is that it presumes that individuals might have *any* preference ordering. Relaxing this requirement is referred to as imposing domain restrictions. Gaertner (2001) provides a comprehensive treatment of the topic. Historically, much of this work was motivated by a desire to establish conditions under which the majority rule core is guaranteed to be nonempty. The consideration of single-peaked domains has resulted in many important results over the past 40 years (for more on these, see Sen & Pattanaik 1969, Blin & Satterthwaite 1976, Moulin 1980, Gaertner 2001, Ballester & Haeringer 2011, Gailmard et al. 2008, Saporiti 2009, Penn et al. 2011, and Duggan 2017). Ferejohn & Grether (1974) and Nakamura (1979) provide an elegant characterization of when restrictions on the number of alternatives will eliminate the possibility of cycles occurring. Finally, Greenberg (1979) and Schofield (1984) consider domain restrictions when the set of alternatives is a compact subset of Euclidean space.

8.4. The Discursive Dilemma and Doctrinal Paradox: Judgment Aggregation

A parallel axiomatic literature with connections to the Arrovian aggregation problem has emerged recently, focused on aggregation of logical conclusions. Just as Arrow's theorem implies that aggregating transitive individual preferences need not generate a transitive result, the doctrinal paradox (Kornhauser & Sager 1986, Kornhauser 1992), also referred to as the discursive dilemma (Pettit 2001), demonstrates that aggregating heterogeneous, internally logically valid conclusions can lead to a logical fallacy. This is a fascinating and active literature, and there are many open questions about the exact relationships between its results and their analogues in social choice theory. We refer the reader to the introductions and reviews provided by List & Puppe (2009) and Grossi & Pigozzi (2014). Reviews of the literature more directly focused on political science have been provided recently by Lax (2011) and Quinn (2012).

9. CONCLUSION

Providing a succinct and unitary conclusion to a review such as this is arguably a hopeless task. Our goal was to illustrate how the literatures that Arrow's theorem has informed and continues to inform are still evolving and, presumably, will continue to evolve far into the future. The heart of Arrow's initial inquiry was measurement, and this job is fundamental, ubiquitous, and—as implied by Arrow's impossibility result—never-ending. That said, Arrow's seemingly negative conclusion is properly seen as a positive result in the sense that it illustrates the power of the axiomatic approach in helping to clarify the important debates about measurement, regardless of the subject of inquiry.

DISCLOSURE STATEMENT

The authors are not aware of any affiliations, memberships, funding, or financial holdings that might be perceived as affecting the objectivity of this review.

ACKNOWLEDGMENTS

We thank Sean Gailmard, Skip Lupia, Dustin Tingley, and the participants at the 2018 Summer Workshop on Empirical Implications of Theoretical Models at the University of Michigan for very helpful comments and suggestions.

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Annual Review of Political Science

Volume 22, 2019

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